INVARIANTS OF KNOTS AND 3-MANIFOLDS FROM QUANTUM GROUPOIDS

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Abstract. We use the categories of representations of finite dimensional quantum groupoids (weak Hopf algebras) to construct ribbon and modular categories that give rise to invariants of knots and 3-manifolds.

Contents

1. Introduction

In [RT2] a general method of constructing invariants of 3-manifolds from modular Hopf algebras was introduced. After appearance of [RT2] it became clear that the technique of Hopf algebras can be replaced by a more general technique of monoidal categories. An appropriate class of categories – modular categories – was introduced in [T1]. In addition to quantum groups, such categories also arise from skein categories of tangles and, as it was observed by A. Ocneanu, from certain bimodule categories of type II_1 subfactors.

The goal of this paper is to study the representation categories of *quantum groupoids* and to give in this way a new construction of modular categories. This extends the construction of modular categories from modular Hopf algebras and in particular from quantum groups at roots of unity.

By quantum groupoids, we understand weak Hopf algebras introduced in [BNSz], [BSz1], [Ni]. These objects generalize Hopf algebras, usual finite groupoid algebras

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and their duals (cf. [NV1]). We use the term "quantum groupoid" rather than "weak Hopf algebra".

It was shown in [NV2], [NV3] that quantum groupoids and their coideal subalgebras are closely related to II_1-subfactors. Every finite index and finite depth II_1-subfactor gives rise to a pair consisting of a C^*-quantum groupoid and its left coideal subalgebra, and vice versa. It was also explained in [NV3] how to express the known subfactor invariants such as bimodule categories and principal graphs in terms of the associated quantum groupoids. In particular, the bimodule categories arising from a finite index and finite depth II_1 subfactors are equivalent to the unitary representation categories of the corresponding C^*-quantum groupoids ([NV3], 5.8).

Thus, it is natural to study categories of representations of quantum groupoids and to extend concepts known for Hopf algebras to this setting. We show that the representation category $\mathrm{Rep}(H)$ of a quantum groupoid H is a monoidal category with duality. We introduce quasitriangular, ribbon, and modular quantum groupoids for which $\mathrm{Rep}(H)$ is, respectively, braided, ribbon, and modular. The notion of factorizability is extended from the Hopf algebra case and used to construct modular categories. We define the Drinfeld double $D(H)$ of a quantum groupoid H and show that it is a factorizable quasitriangular quantum groupoid. For a C^*-quantum groupoid H, we similarly study the unitary representation category $\mathrm{URep}(H)$.

It should be mentioned that the category $\mathrm{URep}(H)$ for a C^*-quantum groupoid H was previously introduced by G. Böhm and K. Szlachányi in [BSz2]; they also introduced the notion of an R-matrix and the Drinfeld double for C^*-quantum groupoids, see [BSz1].

Our main theorem (Theorem 9.8) reads: If H is a connected C^*-quantum groupoid, then the category $\mathrm{URep}(D(H))$ of unitary representations of $D(H)$ is a unitary modular category.

Thus, any finite index and finite depth II_1-subfactor yields a unitary modular category as follows: consider the associated connected C^*-quantum groupoid H, then the category $\mathrm{URep}(D(H))$ is a unitary modular category. We conjecture that this construction is equivalent to the one due to A. Ocneanu (see [EK]). The key role in the proof of the main theorem is played by the following Lemma 8.2: If H is a connected, ribbon and factorizable quantum groupoid with a Haar measure over an algebraically closed field, then $\mathrm{Rep}(H)$ is a modular category.

The organization of the paper is clear from the table of contents.

2. Quantum groupoids

In this section we recall basic properties of quantum groupoids. Most of the material presented here can be found in [BNSz] and [NV1], see also the survey [NV4].

Throughout this paper we use Sweedler's notation for comultiplication, writing $\Delta(b) = b_{(1)} \otimes b_{(2)}$. Let k be a field.

Definition 2.1. A *(finite) quantum groupoid* over k is a finite dimensional k-vector space H with the structures of an associative algebra $(H,\, m,\, 1)$ with multiplication $m : H \otimes_k H \to H$ and unit $1 \in H$ and a coassociative coalgebra $(H,\, \Delta,\, \varepsilon)$ with comultiplication $\Delta : H \to H \otimes_k H$ and counit $\varepsilon : H \to k$ such that:

(i) The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$(\Delta \otimes \mathrm{id})\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1), \tag{1}$$

(ii) The counit is a k-linear map satisfying the identity:

$$\varepsilon(fgh) = \varepsilon(fg_{(1)})\,\varepsilon(g_{(2)}h) = \varepsilon(fg_{(2)})\,\varepsilon(g_{(1)}h), \tag{2}$$

for all $f, g, h \in H$.

(iii) There is an algebra and coalgebra anti-homomorphism $S : H \to H$, called an *antipode*, such that, for all $h \in H$,

$$m(\mathrm{id} \otimes S)\Delta(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \tag{3}$$

$$m(S \otimes \mathrm{id})\Delta(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)). \tag{4}$$

A quantum groupoid is a Hopf algebra if and only if one of the following conditions holds: (i) the comultiplication is unit-preserving or (ii) if and only if the counit is a homomorphism of algebras.

A *morphism* of quantum groupoids is a map between them which is both an algebra and a coalgebra morphism preserving unit and counit and commuting with the antipode. The image of such a morphism is clearly a quantum groupoid. The tensor product of two quantum groupoids is defined in an obvious way.

The set of axioms of Definition 2.1 is self-dual. This allows to define a natural quantum groupoid structure on the dual vector space $\widehat{H} = \mathrm{Hom}_k(H, k)$ by "reversing the arrows":

$$\langle\, h,\, \phi\psi \,\rangle = \langle\, \Delta(h),\, \phi \otimes \psi \,\rangle, \tag{5}$$

$$\langle\, g \otimes h,\, \widehat{\Delta}(\phi) \,\rangle = \langle\, gh,\, \phi \,\rangle, \tag{6}$$

$$\langle\, h,\, \widehat{S}(\phi) \,\rangle = \langle\, S(h),\, \phi \,\rangle, \tag{7}$$

for all $\phi, \psi \in \widehat{H}$, $g, h \in H$. The unit $\widehat{1} \in \widehat{H}$ is ε and counit $\widehat{\varepsilon}$ is $\phi \mapsto \langle\, \phi,\, 1 \,\rangle$.

The linear endomorphisms of H defined by

$$h \mapsto m(\mathrm{id} \otimes S)\Delta(h), \qquad h \mapsto m(S \otimes \mathrm{id})\Delta(h) \tag{8}$$

are called the *target* and *source counital maps* and denoted ε_t and ε_s, respectively. From axioms (3) and (4),

$$\varepsilon_t(h) = (\varepsilon \otimes \mathrm{id})(\Delta(1)(h \otimes 1)), \qquad \varepsilon_s(h) = (\mathrm{id} \otimes \varepsilon)((1 \otimes h)\Delta(1)). \tag{9}$$

In the Hopf algebra case $\varepsilon_t(h) = \varepsilon_s(h) = \varepsilon(h)1$.

We have $S \circ \varepsilon_s = \varepsilon_t \circ S$ and $\varepsilon_s \circ S = S \circ \varepsilon_t$. The images of these maps ε_t and ε_s

$$H_t = \varepsilon_t(H) = \{h \in H \mid \Delta(h) = \Delta(1)(h \otimes 1)\}, \tag{10}$$

$$H_s = \varepsilon_s(H) = \{h \in H \mid \Delta(h) = (1 \otimes h)\Delta(1)\} \tag{11}$$

are subalgebras of H, called the *target* (resp. *source*) *counital subalgebras*. They play the role of ground algebras for H. They commute with each other and

$$H_t = \{(\phi \otimes \mathrm{id})\Delta(1) \mid \phi \in \widehat{H}\}, \qquad H_s = \{(\mathrm{id} \otimes \phi)\Delta(1) \mid \phi \in \widehat{H}\},$$

i.e., H_t (resp. H_s) is generated by the right (resp. left) tensorands of $\Delta(1)$. The restriction of S defines an algebra anti-isomorphism between H_t and H_s. Any non-zero morphism $H \to K$ of quantum groupoids preserves counital subalgebras, i.e., $H_t \cong K_t$ and $H_s \cong K_s$.

In what follows we will use the Sweedler arrows, writing for all $h \in H, \phi \in \widehat{H}$:

$$h \rightharpoonup \phi = \phi_{(1)}\langle h,\, \phi_{(2)}\,\rangle, \qquad \phi \leftharpoonup h = \langle h,\, \phi_{(1)}\,\rangle\phi_{(2)} \tag{12}$$

for all $h \in H, \phi \in \widehat{H}$. Then the map $z \mapsto (z \rightharpoonup \varepsilon)$ is an algebra isomorphism between H_t and $\widehat{H}_s$. Similarly, the map $y \mapsto (\varepsilon \leftharpoonup y)$ is an algebra isomorphism between H_s and $\widehat{H}_t$ ([BNSz], 2.6). Thus, the counital subalgebras of $\widehat{H}$ are canonically anti-isomorphic to those of H.

A quantum groupoid H is called *connected* if $H_s \cap Z(H) = k$, or, equivalently, $H_t \cap Z(H) = k$, where $Z(H)$ denotes the center of H (cf. [N], 3.11, [BNSz], 2.4).

Let us recall that a k-algebra A is *separable* [P] if the multiplication epimorphism $m : A \otimes_k A \to A$ has a right inverse as an $A - A$ bimodule homomorphism. This is equivalent to the existence of a *separability element* $e \in A \otimes_k A$ such that $m(e) = 1$ and $(a \otimes 1)e = e(1 \otimes a),\ (1 \otimes a)e = e(a \otimes 1)$ for all $a \in A$.

The counital subalgebras H_t and H_s are separable, with separability elements $e_t = (S \otimes \mathrm{id})\Delta(1)$ and $e_s = (\mathrm{id} \otimes S)\Delta(1)$, respectively.

Observe that the *adjoint* actions of $1 \in H$ give rise to non-trivial maps $H \to H$:

$$h \mapsto 1_{(1)}hS(1_{(2)}) = \mathrm{Ad}\,^l_1(h), \qquad h \mapsto S(1_{(1)})h1_{(2)} = \mathrm{Ad}\,^r_1(h), \qquad h \in H. \tag{13}$$

Lemma 2.2. *The map Ad^l_1 is a linear projection from H onto $C_H(H_s)$, the centralizer of H_s, i.e., $(Ad^l_1)^2 = Ad^l_1$. The map Ad^r_1 is a linear projection from H onto $C_H(H_t)$, the centralizer of H_t, i.e., $(Ad^r_1)^2 = Ad^r_1$.*

Proof. Since $1_{(1)} \otimes S(1_{(2)})$ is a separability element of H_s, $\mathrm{Ad}\,^l_1(h)$ commutes with H_s. The assertion about $\mathrm{Ad}\,^r_1$ follows similarly. □

Remark 2.3. The opposite algebra H^{op} is also a quantum groupoid with the same coalgebra structure and the antipode S^{-1}. Indeed,

$$\begin{aligned}
S^{-1}(h_{(2)})h_{(1)} &= S^{-1}(\varepsilon_s(h)) = S^{-1}(1_{(1)})\varepsilon(h1_{(2)}) \\
&= S^{-1}(1_{(1)})\varepsilon(hS^{-1}(1_{(2)})) = \varepsilon(h1_{(1)})1_{(2)} = \varepsilon^{op}_t(h), \\
h_{(2)}S^{-1}(h_{(1)}) &= S^{-1}(\varepsilon_t(h)) = \varepsilon(1_{(1)}h)S^{-1}(1_{(2)}) \\
&= \varepsilon(S^{-1}(1_{(1)})h)S^{-1}(1_{(2)}) = 1_{(1)}\varepsilon(1_{(2)}h) = \varepsilon^{op}_s(h), \\
S^{-1}(h_{(3)})h_{(2)}S^{-1}(h_{(1)}) &= S^{-1}(h_{(1)}S(h_{(2)})h_{(3)}) = S^{-1}(h).
\end{aligned}$$

Similarly, the co-opposite coalgebra H^{cop} (with the same algebra structure as H and the opposite coalgebra structure, and the antipode S^{-1}) and $H^{op/cop}$ (with both opposite algebra and coalgebra structures, and the antipode S) are quantum groupoids.

3. Examples of quantum groupoids

Groupoid algebras and their duals ([NV1], 2.1.4). As group algebras and their duals give the simplest examples of Hopf algebras, groupoid algebras and their duals provide simple examples of quantum groupoids.

Let G be a finite *groupoid* (a category with finitely many morphisms, such that each morphism is invertible). Then the groupoid algebra kG (generated by morphisms $g \in G$ with the product of two morphisms being equal to their composition if the latter is defined and 0 otherwise) is a quantum groupoid via :

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \quad g \in G. \tag{14}$$

The counital subalgebras of kG are equal to each other and coincide with the abelian algebra spanned by the identity morphisms : $(kG)_t = (kG)_s = \mathrm{span}\{gg^{-1} \mid g \in G\}$. The target and source counital maps are induced by the operations of taking the target (resp. source) object of a morphism :

$$\varepsilon_t(g) = gg^{-1} = \mathrm{id}_{target(g)} \quad \text{and} \quad \varepsilon_s(g) = g^{-1}g = \mathrm{id}_{source(g)}.$$

The dual quantum groupoid $\widehat{kG}$ is isomorphic to the algebra of functions on G, i.e., it is generated by idempotents p_g, $g \in G$ such that $p_g p_h = \delta_{g,h} p_g$, with

$$\Delta(p_g) = \sum_{uv=g} p_u \otimes p_v, \quad \varepsilon(p_g) = \delta_{g,gg^{-1}}, \quad S(p_g) = p_{g^{-1}}. \tag{15}$$

The target (resp. source) counital subalgebra is precisely the algebra of functions constant on each set of morphisms of G having the same target (resp. source). The target and source maps are

$$\varepsilon_t(p_g) = \sum_{vv^{-1}=g} p_v \quad \text{and} \quad \varepsilon_s(p_g) = \sum_{v^{-1}v=g} p_v.$$

Definition 3.1. We call a quantum groupoid *semisimple* if its underlying algebra is semisimple.

In contrast to finite dimensional semisimple Hopf algebras, the antipode in a finite dimensional quantum groupoid is not necessarily involutive, see Section 9.

Groupoid algebras and their duals give examples of commutative and cocommutative semisimple quantum groupoids.

Quantum transformation groupoids. It is known that any group action on a set (i.e., on a commutative algebra of functions) gives rise to a groupoid [R]. Extending this construction, we associate a quantum groupoid with any action of a Hopf algebra on a separable algebra ("finite quantum space"). Namely, let H be a Hopf algebra and B be a separable (and, therefore, finite dimensional and semisimple [P]) algebra with right H-action $b \otimes h \mapsto b \cdot h$, where $b \in B, h \in H$. Then B^{op}, the algebra opposite to B, becomes a left H-module via $h \otimes a \mapsto h \cdot a = a \cdot S_H(h)$. One can form a *double crossed product algebra* $B^{op} \rtimes H \ltimes B$ with underlying vector space $B^{op} \otimes H \otimes B$ and multiplication

$$(a \otimes h \otimes b)(a' \otimes h' \otimes b') = (h_{(1)} \cdot a')a \otimes h_{(2)} h'_{(1)} \otimes (b \cdot h'_{(2)})b',$$

for all $a, a' \in B^{op}$, $b, b' \in B$, and $h, h' \in H$.

Assume that k is algebraically closed and let e be a separability element of B (note that e is an idempotent when considered in $B^{op} \otimes B$). Let $\omega \in \widehat{B}$ be uniquely determined by $(\omega \otimes \mathrm{id})e = (\mathrm{id} \otimes \omega)e = 1$. One can check that ω is the trace of the left regular representation of B and

$$\omega((h \cdot a)b) = \omega(a(b \cdot h)), \qquad e^{(1)} \otimes (h \cdot e^{(2)}) = (e^{(1)} \cdot h) \otimes e^{(2)},$$

where $a \in B^{op}, b \in B$, and $e = e^{(1)} \otimes e^{(2)}$.

The structure of a quantum groupoid on $B^{op} \rtimes H \ltimes B$ is given by

$$\Delta(a \otimes h \otimes b) = (a \otimes h_{(1)} \otimes e^{(1)}) \otimes ((h_{(2)} \cdot e^{(2)}) \otimes h_{(3)} \otimes b), \tag{16}$$

$$\varepsilon(a \otimes h \otimes b) = \omega(a(h \cdot b)) = \omega(a(b \cdot S_H(h))), \tag{17}$$

$$S(a \otimes h \otimes b) = b \otimes S_H(h) \otimes a. \tag{18}$$

Quantum groupoids $\mathbf{B^{op}\otimes B}$ ([BSz2], 5.2). Let k be algebraically closed and let B be a separable algebra over k, $e = e^{(1)}\otimes e^{(2)} \in B^{op}\otimes B$ be the symmetric separability idempotent of B, and ω be as in the previous example. The map $\pi : x \mapsto e^{(1)}xe^{(2)}$ defines a linear projection from B to $Z(B)$. Let q be an invertible element of B such that $\pi(q) = 1$, then the following operations define a structure of quantum groupoid H_q on $B^{op}\otimes B$:

$$\Delta(b\otimes c) = (b\otimes e^{(1)}q^{-1})\otimes(e^{(2)}\otimes c), \tag{19}$$

$$\varepsilon(b\otimes c) = \omega(qbc), \tag{20}$$

$$S(b\otimes c) = q^{-1}cq\otimes b, \tag{21}$$

for all $b, c \in B$. The target and source counital subalgebras of H_q are $B^{op}\otimes 1$ and $1\otimes B$. The square of the antipode is a conjugation by $g_q = q\otimes q$. Since H_q with different q are non-isomorphic, this example shows that there can be uncountably many non-isomorphic semisimple quantum groupoids with the same underlying algebra (for noncommutative B).

This example can be also explained in terms of *twisting* of quantum groupoids (see [EN], [NV4]).

Quantum groupoids from subfactors. The initial motivation for studying quantum groupoids in [NV1], [NV2], [N] was their connection with depth 2 von Neumann subfactors. This connection was first mentioned in [O] and was also considered in [BNSz], [BSz1], [BSz2], [NSzW]. It was shown in [NV2] that quantum groupoids naturally arise as non-commutative symmetries of subfactors, namely if $N\subset M\subset M_1\subset M_2\subset\dots$ is the Jones tower constructed from a finite index, depth 2 inclusion $N\subset M$ of II_1 factors, then $H = M'\cap M_2$ has a canonical structure of a quantum groupoid acting outerly on M_1 such that $M = M_1^H$ and $M_2 = M_1 \rtimes H$. Furthermore $\widehat{H} = N'\cap M_1$ is a quantum groupoid dual to H.

In [NV3] this result was extended to arbitrary finite depth, via a Galois correspondence and it was shown in ([NV3], 4) that any inclusion of type II_1 von Neumann factors with finite index and depth ([GHJ], 4.1) gives rise to a quantum groupoid and its coideal subalgebra.

We refer the reader to the survey [NV4] (Sections 8,9) and to the Appendix of [NV3] for the explanation of how quantum groupoids can be constructed from subfactors.

Temperley-Lieb algebras. We describe quantum groupoids arising from type A_n subfactors, whose underlying algebras are *Temperley-Lieb algebras* ([GHJ], 2.1).

Let $k = \mathbb{C}$, $\lambda^{-1} = 4\cos^2\frac{\pi}{n+3}$ $(n\geq 2)$, and $e_1, e_2, \dots$ be a sequence of idempotents satisfying, for all i and j, the relations

$$e_ie_{i\pm1}e_i = \lambda e_i,$$

$$e_ie_j = e_je_i, \quad \text{if } |i-j|\geq 2.$$

Let $A_{k,l}$ be the algebra generated by $1, e_k, e_{k+1}, \dots e_l$ $(k\leq l)$, σ be the algebra anti-automorphism of $H = A_{1,2n-1}$ determined by $\sigma(e_i) = e_{2n-i}$ and $P_k \in A_{2n-k,2n-1}\otimes A_{1,k}$ be the image of the separability idempotent of $A_{1,k}$ under $\sigma\otimes\mathrm{id}$.

We denote by τ the non-degenerate Markov trace ([GHJ], 2.1) on H and by w the index of the restriction of τ on $A_{n+1,2n-1}\subset H$, i.e., the unique central element

in $A_{n+1,2n-1}$ such that $\tau(w\,\cdot)$ is equal to the trace of the left regular representation of $A_{n+1,2n-1}$ (see [W]).

Then the following operations define a quantum groupoid structure on H :

$$\begin{aligned}
\Delta(yz) &= (z\otimes y)P_{n-1}, \qquad y\in A_{n+1,2n-1}, \quad z\in A_{1,n-1}\\
\Delta(e_n) &= (1\otimes w)P_n(1\otimes w^{-1}),\\
S(h) &= w^{-1}\sigma(h)w,\\
\varepsilon(h) &= \lambda^{-n}\tau(hfw), \quad h\in A,
\end{aligned}$$

where in the last line

$$f=\lambda^{n(n-1)/2}(e_ne_{n-1}\cdots e_1)(e_{n+1}e_n\cdots e_2)\cdots(e_{2n-1}e_{2n-2}\cdots e_n)$$

is the Jones projection corresponding to the n-step basic construction.

The source and target counital subalgebras of $H=A_{1,2n-1}$ are $H_s=A_{n+1,2n-1}$ and $H_t=A_{1,n-1}$. The example corresponding to $n=2$ is a quantum groupoid of dimension 13 with antipode of infinite order (cf. [NV2], 7.3).

4. Representation category of a quantum groupoid

Throughout this paper we refer to [T2] for definitions related to categories.

For a quantum groupoid H let Rep(H) be the *category of representations* of H, whose objects are finite dimensional left H-modules and whose morphisms are H-linear homomorphisms. We shall show that Rep(H) has a natural structure of a monoidal category with duality.

For objects V,W of Rep(H) set

$$V\otimes W=\{x\in V\otimes_k W \mid x=\Delta(1)\cdot x\}\subset V\otimes_k W\}, \tag{22}$$

with the obvious action of H via the comultiplication Δ (here $\otimes_k$ denotes the usual tensor product of vector spaces). Note that $\Delta(1)$ is an idempotent and therefore $V\otimes W=\Delta(1)\cdot(V\otimes_k W)$. The tensor product of morphisms is the restriction of usual tensor product of homomorphisms. The standard associativity isomorphisms $(U\otimes V)\otimes W\to U\otimes(V\otimes W)$ are functorial and satisfy the pentagon condition, since Δ is coassociative. We will suppress these isomorphisms and write simply $U\otimes V\otimes W$.

The target counital subalgebra $H_t\subset H$ has an H-module structure given by $h\cdot z=\varepsilon_t(hz)$, where $h\in H$, $z\in H_t$.

Lemma 4.1. *H_t is the unit object of Rep(H).*

Proof. Define a k-linear homomorphism $l_V: H_t\otimes V\to V$ by

$$l_V(1_{(1)}\cdot z\otimes 1_{(2)}\cdot v)=z\cdot v, \qquad z\in H_t,\, v\in V.$$

This map is H-linear, since

$$\begin{aligned}
l_V(h\cdot(1_{(1)}\cdot z\otimes 1_{(2)}\cdot v)) &= l_V(h_{(1)}\cdot z\otimes h_{(2)}\cdot v)\\
&= \varepsilon_t(h_{(1)}z)h_{(2)}\cdot v = hz\cdot v\\
&= h\cdot l_V(1_{(1)}\cdot z\otimes 1_{(2)}\cdot v),
\end{aligned}$$

for all $h\in H$. The inverse map $l_V^{-1}: V\to H_t\otimes V$ is given by

$$l_V^{-1}(v)=S(1_{(1)})\otimes(1_{(2)}\cdot v)=(1_{(1)}\cdot 1)\otimes(1_{(2)}\cdot v).$$

The collection $\{l_V\}_V$ gives a natural equivalence between the functor $H_t \otimes (\ )$ and the identity functor. Indeed, for any H-linear homomorphism $f : V \to U$ we have :

$$\begin{aligned} l_U \circ (\mathrm{id} \otimes f)(1_{(1)} \cdot z \otimes 1_{(2)} \cdot v) &= l_U(1_{(1)} \cdot z \otimes 1_{(2)} \cdot f(v)) \\ &= z \cdot f(v) = f(z \cdot v) \\ &= f \circ l_V(1_{(1)} \cdot z \otimes 1_{(2)} \cdot v) \end{aligned}$$

Similarly, the k-linear homomorphism $r_V : V \otimes H_t \to V$ defined by

$$r_V(1_{(1)} \cdot v \otimes 1_{(2)} \cdot z) = S(z) \cdot v, \qquad z \in H_t,\, v \in V,$$

has the inverse $r_V^{-1}(v) = 1_{(1)} \cdot v \otimes 1_{(2)}$ and satisfies the necessary properties.

Finally, we can check the triangle axiom $id_V \otimes l_W = r_V \otimes id_W : V \otimes H_t \otimes W \to V \otimes W$ for all objects V, W of $\mathrm{Rep}(H)$. For $v \in V$, $w \in W$ we have

$$\begin{aligned} &(\mathrm{id}_V \otimes l_W)(1_{(1)} \cdot v \otimes 1'_{(1)} 1_{(2)} \cdot z \otimes 1'_{(2)} \cdot w) = \\ &= 1_{(1)} \cdot v \otimes 1_{(2)} z \cdot w \\ &= 1_{(1)} S(z) \cdot v \otimes 1_{(2)} \cdot w \\ &= (r_V \otimes \mathrm{id}_W)(1'_{(1)} \cdot v \otimes 1'_{(2)} 1_{(1)} \cdot z \otimes 1_{(2)} \cdot w), \end{aligned}$$

therefore, $\mathrm{id}_V \otimes l_W = r_V \otimes \mathrm{id}_W$. □

Using the antipode S of H, we can provide $\mathrm{Rep}(H)$ with a duality. For any object V of $\mathrm{Rep}(H)$, define the action of H on $V^* = \mathrm{Hom}_k(V,\, k)$ by

$$(h \cdot \phi)(v) = \phi(S(h) \cdot v), \tag{23}$$

where $h \in H, v \in V, \phi \in V^*$. For any morphism $f : V \to W$, let $f^* : W^* \to V^*$ be the morphism dual to f (see [T2], I.1.8).

For any V in $\mathrm{Rep}(H)$, we define the duality morphisms

$$d_V : V^* \otimes V \to H_t, \qquad b_V : H_t \to V \otimes V^*$$

as follows. For $\sum_j \phi^j \otimes v_j \in V^* \otimes V$, set

$$d_V(\sum_j \phi^j \otimes v_j) = \sum_j \phi^j(1_{(1)} \cdot v_j) 1_{(2)}. \tag{24}$$

Let $\{f_i\}_i$ and $\{\xi^i\}_i$ be bases of V and V^* respectively, dual to each other. The element $\sum_i f_i \otimes \xi^i$ does not depend on choice of these bases; moreover, for all $v \in V, \phi \in V^*$ one has $\phi = \sum_i \phi(f_i)\xi^i$ and $v = \sum_i f_i \xi^i(v)$. Set

$$b_V(z) = z \cdot (\sum_i f_i \otimes \xi^i). \tag{25}$$

Proposition 4.2. *The category Rep(H) is a monoidal category with duality.*

Proof. We know already that $\mathrm{Rep}(H)$ is monoidal, it remains to prove that d_V and b_V are H-linear and satisfy the identities

$$(\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \qquad (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}.$$

Take $\sum_j \phi^j \otimes v_j \in V^* \otimes V, z \in H_t, h \in H$. Using the axioms of a quantum groupoid, we have

$$\begin{aligned} h \cdot d_V(\sum_j \phi^j \otimes v_j) &= \sum_j \phi^j(1_{(1)} \cdot v)\varepsilon_t(h1_{(2)}) \\ &= \sum_j \phi^j(\varepsilon_s(1_{(1)}h) \cdot v_j)1_{(2)} \\ &= \sum_j \phi^j(S(h_{(1)})1_{(1)}h_{(2)} \cdot v_j)1_{(2)} \\ &= \sum_j (h_{(1)} \cdot \phi^j)(1_{(1)} \cdot (h_{(2)} \cdot v_j))1_{(2)} \\ &= \sum_j d_V(h_{(1)} \cdot \phi^j \otimes h_{(2)} \cdot v_j) \\ &= d_V(h \cdot \sum_j \phi^j \otimes v_j), \end{aligned}$$

therefore, d_V is H-linear. To check the H-linearity of b_V we have to show that $h \cdot b_V(z) = b_V(h \cdot z)$, i.e., that

$$\sum_i h_{(1)} z \cdot f_i \otimes h_{(2)} \cdot \xi^i = \sum_i 1_{(1)}\varepsilon_t(hz) \cdot f_i \otimes 1_{(2)} \cdot \xi^i.$$

Since both sides of the above equality are elements of $V \otimes_k V^*$, evaluating the second factor on $v \in V$, we get the equivalent condition

$$h_{(1)} z S(h_{(2)}) \cdot v = 1_{(1)}\varepsilon_t(hz)S(1_{(2)}) \cdot v,$$

which is easy to check. Thus, b_V is H-linear.

Using the isomorphisms l_V and r_V identifying $H_t \otimes V$, $V \otimes H_t$, and V, for all $v \in V$ and $\phi \in V^*$ we have:

$$\begin{aligned} (\mathrm{id}_V \otimes d_V)(b_V \otimes \mathrm{id}_V)(v) &= (\mathrm{id}_V \otimes d_V)(b_V(1_{(1)} \cdot 1) \otimes 1_{(2)} \cdot v) \\ &= (\mathrm{id}_V \otimes d_V)(b_V(1_{(2)}) \otimes S^{-1}(1_{(1)}) \cdot v) \\ &= \sum_i (\mathrm{id}_V \otimes d_V)(1_{(2)} \cdot f_i \otimes 1_{(3)} \cdot \xi^i \otimes S^{-1}(1_{(1)}) \cdot v) \\ &= \sum_i 1_{(2)} \cdot f_i \otimes (1_{(3)} \cdot \xi^i)(1'_{(1)}S^{-1}(1_{(1)}) \cdot v)1'_{(2)} \\ &= 1_{(2)}S(1_{(3)})1'_{(1)}S^{-1}(1_{(1)}) \cdot v \otimes 1'_{(2)} = v, \\ (d_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes b_V)(\phi) &= (d_V \otimes \mathrm{id}_{V^*})(1_{(1)} \cdot \phi \otimes b_V(1_{(2)})) \\ &= \sum_i (d_V \otimes \mathrm{id}_{V^*})(1_{(1)} \cdot \phi \otimes 1_{(2)} \cdot f_i \otimes 1_{(3)} \cdot \xi^i) \\ &= \sum_i (1_{(1)} \cdot \phi)(1'_{(1)}1_{(2)} \cdot f_i)1'_{(2)} \otimes 1_{(3)} \cdot \xi^i \\ &= 1'_{(2)} \otimes 1_{(3)}1_{(1)}S(1'_{(1)}1_{(2)}) \cdot \phi = \phi, \end{aligned}$$

which completes the proof. □

Remark 4.3. Similarly to the construction of $\mathrm{Rep}(H)$, one can construct a category of right H-modules, in which H_s plays the role of the unit object.

5. Quasitriangular quantum groupoids

Definition 5.1. A quasitriangular quantum groupoid is a pair $(H, \mathcal{R})$ where H is a quantum groupoid and $\mathcal{R} \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)$ satisfying the following conditions :

$$\Delta^{op}(h)\mathcal{R} = \mathcal{R}\Delta(h), \tag{26}$$

for all $h \in H$, where Δ^{op} denotes the comultiplication opposite to Δ,

$$(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \qquad (\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{27}$$

where $\mathcal{R}_{12} = \mathcal{R} \otimes 1$, $\mathcal{R}_{23} = 1 \otimes \mathcal{R}$, etc. as usual, and such that there exists $\bar{\mathcal{R}} \in \Delta(1)(H \otimes_k H)\Delta^{op}(1)$ with

$$\mathcal{R}\bar{\mathcal{R}} = \Delta^{op}(1), \qquad \bar{\mathcal{R}}\mathcal{R} = \Delta(1). \tag{28}$$

Note that $\bar{\mathcal{R}}$ is uniquely determined by $\mathcal{R}$: if $\bar{\mathcal{R}}$ and $\bar{\mathcal{R}}'$ are two elements of $\Delta(1)(H \otimes_k H)\Delta^{op}(1)$ satisfying the previous equation, then

$$\bar{\mathcal{R}} = \bar{\mathcal{R}}\Delta^{op}(1) = \bar{\mathcal{R}}\mathcal{R}\bar{\mathcal{R}}' = \Delta(1)\bar{\mathcal{R}}' = \bar{\mathcal{R}}'.$$

For any two objects V and W of $\mathrm{Rep}(H)$ define $c_{V,W} : V \otimes W \to W \otimes V$ as the action of $\mathcal{R}_{21}$:

$$c_{V,W}(x) = \mathcal{R}^{(2)} \cdot x^{(2)} \otimes \mathcal{R}^{(1)} \cdot x^{(1)}, \tag{29}$$

where $x = x^{(1)} \otimes x^{(2)} \in V \otimes W$, $\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)$.

Proposition 5.2. *The family of homomorphisms $\{c_{V,W}\}_{V,W}$ defines a braiding in Rep(H). Conversely, if Rep(H) is braided, then there exists $\mathcal{R} \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)$, satisfying the properties of Definition 5.1 and inducing the given braiding.*

Proof. Note that $c_{V,W}$ is well-defined, since $\mathcal{R}_{21} = \Delta(1)\mathcal{R}_{21}$. To prove the H-linearity of $c_{V,W}$ we observe that

$$\begin{aligned} c_{V,W}(h \cdot x) &= \mathcal{R}^{(2)}h_{(2)} \cdot x^{(2)} \otimes \mathcal{R}^{(1)}h_{(1)} \cdot x^{(1)} \\ &= h_{(1)}\mathcal{R}^{(2)} \cdot x^{(2)} \otimes h_{(2)}\mathcal{R}^{(1)} \cdot x^{(1)} = h \cdot (c_{V,W}(x)). \end{aligned}$$

The inverse of $c_{V,W}$ is given by

$$c^{-1}_{V,W}(y) = \bar{\mathcal{R}}^{(1)} \cdot y^{(2)} \otimes \bar{\mathcal{R}}^{(2)} \cdot y^{(1)},$$

where $y = y^{(1)} \otimes y^{(2)} \in W \otimes V, \bar{\mathcal{R}} = \bar{\mathcal{R}}^{(1)} \otimes \bar{\mathcal{R}}^{(2)}$. Therefore, $c_{V,W}$ is an isomorphism. Finally, one can verify that the braiding identities

$$(\mathrm{id}_V \otimes c_{U,W})(c_{U,V} \otimes id_W) = c_{U,V\otimes W}, \qquad (c_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c_{V,W}) = c_{U\otimes V,W}$$

are equivalent to the relations of Definition 5.1, exactly in the same way as in the case of Hopf algebras (see, for instance, ([T2], XI, 2.3.1)). □

Lemma 5.3. *Let $(H, \mathcal{R})$ be a quasitriangular quantum groupoid. Then for all $y \in H_s, z \in H_t$ the following six identities hold:*

$$\begin{aligned} (1 \otimes z)\mathcal{R} &= \mathcal{R}(z \otimes 1), & (y \otimes 1)\mathcal{R} &= \mathcal{R}(1 \otimes y), \\ (z \otimes 1)\mathcal{R} &= (1 \otimes S(z))\mathcal{R}, & (1 \otimes y)\mathcal{R} &= (S(y) \otimes 1)\mathcal{R}, \\ \mathcal{R}(y \otimes 1) &= \mathcal{R}(1 \otimes S(y)), & \mathcal{R}(1 \otimes z) &= \mathcal{R}(S(z) \otimes 1). \end{aligned}$$

Proof. Since we have $\Delta^{op}(1)\mathcal{R} = \mathcal{R} = \mathcal{R}\Delta(1)$, the first line is a consequence of the relation $\Delta(yz) = (z \otimes y)\Delta(1)$, the second line follows from $(1 \otimes z)\Delta(1) = (S(z) \otimes 1)\Delta(1)$ and $(y \otimes 1)\Delta(1) = (1 \otimes S(y))\Delta(1)$. The last two identities are proven similarly. □

Proposition 5.4. *Let $(H, \mathcal{R})$ be a quasitriangular quantum groupoid. Then $\mathcal{R}$ satisfies the quantum Yang-Baxter equation :*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Proof. It follows from the first two relations of Definition 5.1, that

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = (\mathrm{id} \otimes \Delta^{op})(\mathcal{R})\mathcal{R}_{23} = \mathcal{R}_{23}(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

□

Remark 5.5. Let us define two k-linear maps $\mathcal{R}_1, \mathcal{R}_2 : \widehat{H} \to H$ by

$$\mathcal{R}_1(\phi) = (\mathrm{id} \otimes \phi)(\mathcal{R}), \quad \mathcal{R}_2(\phi) = (\phi \otimes \mathrm{id})(\mathcal{R}), \quad \text{for } \phi \in \widehat{H}.$$

Then the condition $(\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ is equivalent to $\mathcal{R}_1$ being a coalgebra homomorphism and algebra anti-homomorphism and the condition $(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ is equivalent to $\mathcal{R}_2$ being an algebra homomorphism and coalgebra anti-homomorphism. In other words, $\mathcal{R}_1$ and $\mathcal{R}_2$ are homomorphisms of quantum groupoids $\widehat{H} \to H^{op}$ and $\widehat{H} \to H^{cop}$, respectively.

Proposition 5.6. *For any quasitriangular quantum groupoid $(H, \mathcal{R})$, we have:*

$$(\varepsilon_s \otimes id)(\mathcal{R}) = \Delta(1), \qquad (id \otimes \varepsilon_s)(\mathcal{R}) = (S \otimes id)\Delta^{op}(1),$$
$$(\varepsilon_t \otimes id)(\mathcal{R}) = \Delta^{op}(1), \qquad (id \otimes \varepsilon_t)(\mathcal{R}) = (S \otimes id)\Delta(1),$$
$$(S \otimes id)(\mathcal{R}) = (id \otimes S^{-1})(\mathcal{R}) = \bar{\mathcal{R}}, \; (S \otimes S)(\mathcal{R}) = \mathcal{R}.$$

Proof. First, using the same argument as in [T2], XI, 2.1.1, we can show that $(\varepsilon \otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id} \otimes \varepsilon)(\mathcal{R}) = 1$. Next, using Lemma 5.3, we obtain

$$\begin{aligned}
(\varepsilon_s \otimes \mathrm{id})(\mathcal{R}) &= 1_{(1)}\varepsilon(\mathcal{R}^{(1)}1_{(2)}) \otimes \mathcal{R}^{(2)} \\
&= 1_{(1)}\varepsilon(\mathcal{R}^{(1)}) \otimes 1_{(2)}\mathcal{R}^{(2)} = \Delta(1), \\
(\mathrm{id} \otimes \varepsilon_s)(\mathcal{R}) &= \mathcal{R}^{(1)} \otimes 1_{(1)}\varepsilon(\mathcal{R}^{(2)}1_{(2)}) \\
&= S(1_{(2)})\mathcal{R}^{(1)} \otimes 1_{(1)}\varepsilon(\mathcal{R}^{(2)}) = (S \otimes \mathrm{id})\Delta^{op}(1), \\
(\varepsilon_t \otimes \mathrm{id})(\mathcal{R}) &= \varepsilon(1_{(1)}\mathcal{R}^{(1)})1_{(2)} \otimes \mathcal{R}^{(2)} \\
&= \varepsilon(\mathcal{R}^{(1)})1_{(2)} \otimes \mathcal{R}^{(2)}1_{(1)} = \Delta^{op}(1), \\
(\mathrm{id} \otimes \varepsilon_t)(\mathcal{R}) &= \mathcal{R}^{(1)} \otimes \varepsilon(1_{(1)}\mathcal{R}^{(2)})1_{(2)} \\
&= S(1_{(1)})\mathcal{R}^{(1)} \otimes \varepsilon(\mathcal{R}^{(2)})1_{(2)} = (S \otimes \mathrm{id})\Delta(1).
\end{aligned}$$

Let m denote multiplication $H \otimes_k H \to H$ in H. Set $m_{12} = m \otimes \mathrm{id} : H^{\otimes 3} \to H^{\otimes 2}$ and $m_{23} = \mathrm{id} \otimes m : H^{\otimes 3} \to H^{\otimes 2}$. It follows from the above relations that

$$\begin{aligned}
m_{12}((S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R})) &= (\varepsilon_s \otimes \mathrm{id})(\mathcal{R}) = \Delta(1), \\
m_{12}((\mathrm{id} \otimes S \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(\mathcal{R})) &= (\varepsilon_t \otimes \mathrm{id})(\mathcal{R}) = \Delta^{op}(1), \\
m_{23}((\mathrm{id} \otimes \mathrm{id} \otimes S^{-1})(\Delta^{op} \otimes \mathrm{id})(\mathcal{R})) &= (\mathrm{id} \otimes S^{-1}\varepsilon_t)(\mathcal{R}) = \Delta^{op}(1), \\
m_{23}((\mathrm{id} \otimes S^{-1} \otimes \mathrm{id})(\Delta^{op} \otimes \mathrm{id})(\mathcal{R})) &= (\mathrm{id} \otimes S^{-1}\varepsilon_s)(\mathcal{R}) = \Delta(1).
\end{aligned}$$

On the other hand, Definition 5.1 implies that

$$\begin{aligned}
& m_{12}((S\otimes \mathrm{id}\otimes \mathrm{id})(\Delta\otimes \mathrm{id})(\mathcal{R})) = \\
& \quad = \quad m_{12}((S\otimes \mathrm{id}\otimes \mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})) = (S\otimes \mathrm{id})(\mathcal{R})\mathcal{R}, \\
& m_{12}((\mathrm{id}\otimes S\otimes \mathrm{id})(\Delta\otimes \mathrm{id})(\mathcal{R})) = \\
& \quad = \quad m_{12}((\mathrm{id}\otimes S\otimes \mathrm{id})(\mathcal{R}_{13}\mathcal{R}_{23})) = \mathcal{R}(S\otimes \mathrm{id})(\mathcal{R}), \\
& m_{23}((\mathrm{id}\otimes \mathrm{id}\otimes S^{-1})(\Delta^{op}\otimes \mathrm{id})(\mathcal{R})) = \\
& \quad = \quad m_{23}((\mathrm{id}\otimes \mathrm{id}\otimes S^{-1})(\mathcal{R}_{12}\mathcal{R}_{13})) = \mathcal{R}(\mathrm{id}\otimes S^{-1})(\mathcal{R}), \\
& m_{23}((\mathrm{id}\otimes S^{-1}\otimes \mathrm{id})(\Delta^{op}\otimes \mathrm{id})(\mathcal{R})) = \\
& \quad = \quad m_{23}((\mathrm{id}\otimes S^{-1}\otimes \mathrm{id})(\mathcal{R}_{12}\mathcal{R}_{13})) = (\mathrm{id}\otimes S^{-1})(\mathcal{R})\mathcal{R}.
\end{aligned}$$

Therefore, $(S\otimes \mathrm{id})(\mathcal{R}) = (\mathrm{id}\otimes S^{-1})(\mathcal{R}) = \bar{\mathcal{R}}$ and $(S\otimes S)(\mathcal{R}) = \mathcal{R}$. □

Proposition 5.7. *Let $(H,\mathcal{R})$ be a quasitriangular quantum groupoid. Then $S^2(h) = uhu^{-1}$ for all $h\in H$, where $u = S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ is an invertible element of H such that*

$$u^{-1} = \mathcal{R}^{(2)}S^2(\mathcal{R}^{(1)}), \quad \Delta(u) = \bar{\mathcal{R}}\bar{\mathcal{R}}_{21}(u\otimes u).$$

Likewise, $S^{-2}(h) = vhv^{-1}$, where $v = S(u) = \mathcal{R}^{(1)}S(\mathcal{R}^{(2)})$, and

$$v^{-1} = S^2(\mathcal{R}^{(1)})\mathcal{R}^{(2)}, \quad \Delta(v) = \bar{\mathcal{R}}\bar{\mathcal{R}}_{21}(v\otimes v).$$

Proof. Note that $S(\mathcal{R}^{(2)})y\mathcal{R}^{(1)} = S(y)u$ for all $y\in H_s$, by Lemma 5.3. Hence, we have

$$\begin{aligned}
S(h_{(2)})uh_{(1)} &= S(h_{(2)})S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}h_{(1)} = S(\mathcal{R}^{(2)}h_{(2)})\mathcal{R}^{(1)}h_{(1)} \\
&= S(h_{(1)}\mathcal{R}^{(2)})h_{(2)}\mathcal{R}^{(1)} = S(\mathcal{R}^{(2)})\varepsilon_s(h)\mathcal{R}^{(1)} \\
&= S(\varepsilon_s(h))u,
\end{aligned}$$

for all $h\in H$. Therefore, using the axioms of a quantum groupoid, we get

$$\begin{aligned}
uh &= S(1_{(2)})u1_{(1)}h = S(\varepsilon_t(h_{(2)})uh_{(1)} \\
&= S(h_{(2)}S(h_{(3)}))uh_{(1)} = S^2(h_{(3)})S(h_{(2)})uh_{(1)} \\
&= S^2(h_{(2)})S(\varepsilon_s(h_{(1)}))u = S(\varepsilon_s(h_{(1)})S(h_{(2)}))u = S^2(h)u.
\end{aligned}$$

The remaining part of the proof follows the lines of ([M], 2.1.8). The results for v can be obtained by applying the results for u to the quasitriangular quantum groupoid $(H^{op/cop},\mathcal{R})$. □

Definition 5.8. The element u defined in Proposition 5.7 is called *the Drinfeld element* of H.

Corollary 5.9. *The element $uv = vu$ is central and*

$$\Delta(uv) = (\bar{\mathcal{R}}\bar{\mathcal{R}}_{21})^2(uv\otimes uv).$$

The element $uv^{-1} = vu^{-1}$ is group-like and $S^4(h) = uv^{-1}hvu^{-1}$ for all $h\in H$.

Proof. See ([M], 2.1.9). □

Proposition 5.10. *Given a quasitriangular quantum groupoid* $(H, \mathcal{R})$*, consider a linear map* $F : \widehat{H} \to H$ *given by*

$$F(\phi) = (\phi \otimes id)(\mathcal{R}_{21}\mathcal{R}), \qquad \phi \in \widehat{H}. \tag{30}$$

Then the image of F *lies in* $C_H(H_s)$*, the centralizer of* H_s*.*

Proof. Take $y \in H_s$. Then we have

$$\phi(\mathcal{R}^{(2)}\mathcal{R}^{(1)})\mathcal{R}^{(1)}\mathcal{R}^{(2)}y = \phi(\mathcal{R}^{(2)}y\mathcal{R}^{(1)})\mathcal{R}^{(1)}\mathcal{R}^{(2)} = \phi(\mathcal{R}^{(2)}\mathcal{R}^{(1)})y\mathcal{R}^{(1)}\mathcal{R}^{(2)}.$$

Therefore $F(\phi) \in C_H(H_s)$, as required. □

Definition 5.11 (cf. [M], 2.1.12)**.** A quasitriangular quantum groupoid is *factorizable* if the map $F : \widehat{H} \to C_H(H_s)$ from Proposition 5.10 is surjective.

The factorizability means that $\mathcal{R}$ is as non-trivial as possible, in contrast to *triangular* quantum groupoids, for which $\bar{\mathcal{R}} = \mathcal{R}_{21}$ and $F(\widehat{H}) = H_t$.

Corollary 5.12. *If* H *is factorizable, then the restriction of* F *to the subspace* $W_s = \{\phi \in \widehat{H} \mid \phi = \phi \circ Ad_1^r)\}$ *is a linear isomorphism onto* $C_H(H_s)$*.*

Proof. From the observation that $F(\phi) = F(\phi \circ \mathrm{Ad}\,_1^r)$ we have that the restriction of F to W_s is a linear map onto $C_H(H_s)$. On the other hand, Lemma 2.2 allows to identify W_s with the dual vector space to $C_H(H_t)$, from where $\dim W_s = \dim C_H(H_s)$ and the result follows. □

6. The Drinfeld double for quantum groupoids

Let H be a finite quantum groupoid. We define the *Drinfeld double* $D(H)$ of H as follows. Consider on the vector space $\widehat{H}^{op} \otimes_k H$ a multiplication given by

$$(\phi \otimes h)(\psi \otimes g) = \psi_{(2)}\phi \otimes h_{(2)}g\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle, \tag{31}$$

where $\phi, \psi \in \widehat{H}^{op}$ and $h, g \in H$. We verify below that the linear span J of the elements

$$\phi \otimes zh \quad - \quad (\varepsilon \leftharpoonup z)\phi \otimes h, \quad z \in H_t, \tag{32}$$
$$\phi \otimes yh \quad - \quad (y \rightharpoonup \varepsilon)\phi \otimes h, \quad y \in H_s, \tag{33}$$

is a two-sided ideal in $\widehat{H}^{op} \otimes_k H$. Let $D(H)$ be the factor-algebra $(\widehat{H}^{op} \otimes_k H)/J$ and let $[\phi \otimes h]$ denote the class of $\phi \otimes h$ in $D(H)$.

Definition and Theorem 6.1. $D(H)$ is a quantum groupoid with unit $[\varepsilon \otimes 1]$, and comultiplication, counit, and antipode given by

$$\Delta([\phi \otimes h]) \;=\; [\phi_{(1)} \otimes h_{(1)}] \otimes [\phi_{(2)} \otimes h_{(2)}], \tag{34}$$
$$\varepsilon([\phi \otimes h]) \;=\; \langle\, \varepsilon_t(h),\, \phi\,\rangle, \tag{35}$$
$$S([\phi \otimes h]) \;=\; [S^{-1}(\phi_{(2)}) \otimes S(h_{(2)})]\langle\, h_{(1)},\, \phi_{(1)}\,\rangle\langle\, S(h_{(3)}),\, \phi_{(3)}\,\rangle. \tag{36}$$

In the case where H is a Hopf algebra, this definition and is due to Drinfeld [D].

Proof. Associativity of multiplication in $\widehat{H}^{op} \otimes_k H$ and hence in $D(H)$ can be verified exactly as in ([M], 7.1.1). Let us check that J is an ideal. We have :

$$\begin{aligned}
(\phi \otimes h)((\varepsilon \leftharpoonup z)\psi \otimes g) &= \psi_{(2)}\phi \otimes h_{(2)}g\langle\, S(h_{(1)}),\, (\varepsilon \leftharpoonup z)\psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle \\
&= \psi_{(2)}\phi \otimes h_{(3)}g\langle\, zS(h_{(2)}),\, \varepsilon\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(4)},\, \psi_{(3)}\,\rangle \\
&= \psi_{(2)}\phi \otimes h_{(2)}zg\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle \\
&= (\phi \otimes h)(\psi \otimes zg), \\
(\psi \otimes zg)(\phi \otimes h) &= \phi_{(2)}\psi \otimes g_{(2)}h\langle\, S(zg_{(1)}),\, \phi_{(1)}\,\rangle\langle\, g_{(3)},\, \phi_{(3)}\,\rangle \\
&= \langle\, Sz,\, \phi_{(2)}\,\rangle\phi_{(3)}\psi \otimes g_{(2)}h\langle\, S(g_{(1)}),\, \phi_{(1)}\,\rangle\langle\, g_{(3)},\, \phi_{(3)}\,\rangle \\
&= \phi_{(2)}(\varepsilon \leftharpoonup z)\psi \otimes g_{(2)}h\langle\, S(g_{(1)}),\, \phi_{(1)}\,\rangle\langle\, g_{(3)},\, \phi_{(3)}\,\rangle \\
&= ((\varepsilon \leftharpoonup z)\psi \otimes g)(\phi \otimes h),
\end{aligned}$$

where $z \in H_t$ and we used the identity $zS(h_{(1)}) \otimes h_{(2)} = S(h_{(1)}) \otimes h_{(2)}z$. Similarly, one checks that

$$\begin{aligned}
(\psi \otimes yg)(\phi \otimes h) &= ((y \rightharpoonup \varepsilon)\psi \otimes g)(\phi \otimes h), \\
(\phi \otimes h)(\psi \otimes yg) &= (\phi \otimes h)((y \rightharpoonup \varepsilon)\psi \otimes g),
\end{aligned}$$

therefore for all $x \in J$ we have $(\phi \otimes h)x = x(\phi \otimes h) = 0$, so J is an ideal. We also compute

$$\begin{aligned}
[\varepsilon \otimes 1][\phi \otimes h] &= [\phi_{(2)} \otimes 1_{(2)}h\langle\, S(1_{(1)}),\, \phi_{(1)}\,\rangle\langle\, 1_{(3)},\, \phi_{(3)}\,\rangle] \\
&= [\phi_{(2)} \otimes \langle\, S(1_{(1)}),\, \phi_{(1)}\,\rangle 1_{(2)}1'_{(1)}\langle\, 1'_{(2)},\, \phi_{(3)}\,\rangle h] \\
&= [\varepsilon_t(\phi_{(1)})S(\varepsilon_t(\phi_{(3)}))\phi_{(2)} \otimes h] = [\phi \otimes h],
\end{aligned}$$

and similarly $[\phi \otimes h][\varepsilon \otimes 1] = [\phi \otimes h]$, so that $[\varepsilon \otimes 1]$ is a unit.

Now let us verify that the structure maps Δ, ε, and S are well-defined on $D(H)$. We have, using properties of a quantum groupoid and its counital subalgebras:

$$\begin{aligned}
\Delta([\phi \otimes zh]) &= [\phi_{(1)} \otimes zh_{(1)}] \otimes [\phi_{(2)} \otimes h_{(2)}] \\
&= [(\varepsilon \leftharpoonup z)\phi_{(1)} \otimes h_{(1)}] \otimes [\phi_{(2)} \otimes h_{(2)}] \\
&= \Delta([\langle\, z,\, \varepsilon_{(1)}\,\rangle\varepsilon_{(2)}\phi \otimes h]), \\
\varepsilon([\langle\, z,\, \varepsilon_{(1)}\,\rangle\varepsilon_{(2)}\phi \otimes h]) &= \langle\, z,\, \varepsilon_{(1)}\,\rangle\langle\, \varepsilon_t(h),\, \varepsilon_{(2)}\phi\,\rangle \\
&= \langle\, z,\, \varepsilon_{(1)}\,\rangle\langle\, 1_{(1)}\varepsilon_t(h),\, \varepsilon_{(2)}\,\rangle\langle\, 1_{(2)},\, \phi\,\rangle \\
&= \langle\, z\varepsilon_t(h)1_{(1)},\, \varepsilon\,\rangle\langle\, 1_{(2)},\, \phi\,\rangle \\
&= \langle\, z\varepsilon_t(h),\, \phi\,\rangle = \varepsilon([\phi \otimes zh]), \\
S([(\varepsilon \leftharpoonup z)\phi \otimes h]) &= [\langle\, z,\, \varepsilon_{(1)}\,\rangle S^{-1}(\varepsilon_{(3)}\phi_{(2)}) \otimes S(h_{(2)})] \\
&\quad \langle\, h_{(1)},\, \varepsilon_{(2)}\phi_{(1)}\,\rangle\langle\, S(h_{(3)}),\, \varepsilon_{(4)}\phi_{(3)}\,\rangle \\
&= [S^{-1}(\phi_{(2)}) \otimes S(h_{(2)})] \\
&\quad \langle\, h_{(1)},\, (\varepsilon \leftharpoonup z)\phi_{(1)}\,\rangle\langle\, S(h_{(3)}),\, \phi_{(3)}\,\rangle \\
&= [S^{-1}(\phi_{(2)}) \otimes S(h_{(3)})]\langle\, zh_{(1)},\, \varepsilon\,\rangle \\
&\quad \langle\, h_{(2)},\, \phi_{(1)}\,\rangle\langle\, S(h_{(4)}),\, \phi_{(3)}\,\rangle \\
&= [S^{-1}(\phi_{(2)}) \otimes S(h_{(2)})]\langle\, zh_{(1)},\, \phi_{(1)}\,\rangle\langle\, S(h_{(3)}),\, \phi_{(3)}\,\rangle \\
&= S([\phi \otimes zh])
\end{aligned}$$

for all $h \in H, \phi \in \widehat{H}, z \in H_t$. Next, we need to check the axioms of a quantum groupoid. Coassociativity and multiplicativity of Δ are established as in ([M],

7.1.1), since the computations given there do not use the unitality of multiplication and comultiplication. For the counit property, we have:

$$\begin{aligned}
(\varepsilon\otimes \mathrm{id})\Delta([\phi\otimes h]) &= \langle\, \varepsilon_t(h_{(1)}),\, \phi_{(1)}\,\rangle[\phi_{(2)}\otimes h_{(2)}]\\
&= \langle\, \varepsilon_t(h_{(1)}),\, \varepsilon_{(1)}\,\rangle[\varepsilon_{(2)}\phi\otimes h_{(2)}]\\
&= [\phi\otimes \varepsilon_t(h_{(1)})h_{(2)}] = [\phi\otimes h],\\
(\mathrm{id}\otimes\varepsilon)\Delta([\phi\otimes h]) &= [\phi_{(1)}\otimes h_{(1)}]\langle\, \varepsilon_t(h_{(2)}),\, \phi_{(2)}\,\rangle\\
&= [\phi_{(1)}\otimes 1_{(1)}h]\langle\, 1_{(2)},\, \varepsilon_t(\phi_{(2)})\,\rangle\\
&= [S^{-1}(\varepsilon_t(\phi_{(2)}))\phi_{(1)}\otimes h] = [\phi\otimes h],
\end{aligned}$$

where we used the amalgamation property $[\phi\otimes zh] = [(\varepsilon \leftharpoonup z)\phi\otimes h], \quad z\in H_t$, following from (32). Now we verify the remaining axioms of a quantum groupoid. For all $h, g, f\in H$ and $\phi,\psi,\theta\in \widehat{H}$ we compute

$$\begin{aligned}
&\varepsilon([\phi\otimes h][\psi\otimes g][\theta\otimes f]) =\\
&\quad= \varepsilon([\theta_{(2)}\psi_{(2)}\phi\otimes h_{(3)}g_{(2)}f])\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\\
&\qquad \langle\, S(h_{(2)}g_{(1)}),\theta_{(1)}\,\rangle\langle\, h_{(4)}g_{(3)},\, \theta_{(3)}\,\rangle\langle\, h_{(5)},\, \psi_{(3)}\,\rangle\\
&\quad= \langle\, h_{(3)}g_{(2)},\, \varepsilon_{(1)}\varepsilon_t(\theta_{(2)}\psi_{(2)}\phi)\,\rangle\langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\\
&\qquad \langle\, h_{(2)}g_{(1)},\, S(\theta_{(1)})\,\rangle\langle\, h_{(4)}g_{(3)},\, \theta_{(3)}\,\rangle\langle\, h_{(5)},\, \psi_{(3)}\,\rangle\\
&\quad= \langle\, h_{(2)}g,\, S(\theta_{(1)})\varepsilon_{(1)}\varepsilon_t(\theta_{(2)}\psi_{(2)}\phi)\theta_{(3)}\,\rangle\langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle\\
&\quad= \langle\, h_{(2)}g,\, \varepsilon_t(\psi_{(2)}\phi)\varepsilon_s(\varepsilon_{(1)}\theta)\,\rangle\langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle\\
&\quad= \langle\, h_{(2)}g_{(1)},\, \varepsilon_t(\psi_{(2)}\phi)\,\rangle\langle\, g_{(2)},\, \varepsilon'_{(2)}\varepsilon_s(\varepsilon_{(1)}\theta)\,\rangle\\
&\qquad \langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \varepsilon'_{(1)}\psi_{(3)}\,\rangle\\
&\quad= \langle\, h_{(2)}g_{(1)},\, \varepsilon_t(\psi_{(2)}\phi)\,\rangle\langle\, g_{(2)},\, S(\theta_{(1)})\varepsilon_{(1)}\varepsilon_t(\theta_{(2)}\psi_{(4)})\theta_{(3)}\,\rangle\\
&\qquad \langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \varepsilon'_{(1)}\psi_{(3)}\,\rangle\\
&\quad= \langle\, h_{(2)}g_{(1)},\, \varepsilon_t(\psi_{(2)}\phi)\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle\\
&\qquad \langle\, g_{(3)}f,\, \varepsilon_t(\theta_{(2)}\psi_{(4)})\,\rangle\langle\, S(g_{(2)}),\, \theta_{(1)}\,\rangle\langle\, g_{(4)},\, \theta_{(3)}\,\rangle\\
&\quad= \varepsilon([\phi\otimes h][\psi_{(1)}\otimes g_{(1)}])\varepsilon([\psi_{(2)}\otimes g_{(2)}][\theta\otimes f]),\\
&\varepsilon([\phi\otimes h][\psi_{(2)}\otimes g_{(2)}])\varepsilon([\psi_{(1)}\otimes g_{(1)}][\theta\otimes f]) =\\
&\quad= \langle\, h_{(2)}g_{(4)},\, \varepsilon_t(\psi_{(3)}\phi)\,\rangle\langle\, S(h_{(1)}),\, \psi_{(2)}\,\rangle\langle\, h_{(3)},\, \psi_{(4)}\,\rangle\\
&\qquad \langle\, g_{(2)}f,\, \varepsilon_t(\theta_{(2)}\psi_{(1)})\,\rangle\langle\, S(g_{(1)}),\, \theta_{(1)}\,\rangle\langle\, g_{(3)},\, \theta_{(3)}\,\rangle\\
&\quad= \langle\, h_{(2)}g_{(2)},\, \varepsilon_t(\psi_{(3)}\phi)\,\rangle\langle\, g_{(1)},\, S(\theta_{(1)})\varepsilon_{(1)}\varepsilon_t(\theta_{(2)}\psi_{(1)})\theta_{(3)}\,\rangle\\
&\qquad \langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(2)}\,\rangle\langle\, h_{(3)},\, \psi_{(4)}\,\rangle\\
&\quad= \langle\, h_{(2)}g_{(2)},\, \varepsilon_t(\psi_{(3)}\phi)\,\rangle\langle\, g_{(1)},\, \varepsilon'_{(2)}\varepsilon_s(\varepsilon_{(1)}\theta)\,\rangle\\
&\qquad \langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, S(\varepsilon'_{(1)})\psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(4)}\,\rangle\\
&\quad= \langle\, h_{(2)}g,\, \varepsilon_t(\psi_{(2)}\phi)\varepsilon_s(\varepsilon_{(1)}\theta)\,\rangle\langle\, f,\, \varepsilon_{(2)}\,\rangle\langle\, S(h_{(1)}),\, \psi_{(1)}\,\rangle\langle\, h_{(3)},\, \psi_{(3)}\,\rangle\\
&\quad= \varepsilon([\phi\otimes h][\psi\otimes g][\theta\otimes f]),
\end{aligned}$$

which is axiom (2). For axiom (1) we have:

$$\begin{aligned}
&(\Delta([\varepsilon\otimes 1])\otimes[\varepsilon\otimes 1])([\varepsilon\otimes 1]\otimes\Delta([\varepsilon\otimes 1])) =\\
&\quad= [\varepsilon_{(1)}\otimes 1_{(1)}]\otimes[\varepsilon_{(2)}\otimes 1_{(2)}][\varepsilon'_{(1)}\otimes 1'_{(1)}]\otimes[\varepsilon'_{(2)}\otimes 1'_{(2)}]\\
&\quad= [\varepsilon_{(1)}\otimes 1_{(1)}]\otimes[\varepsilon'_{(1)}\varepsilon_{(2)}\otimes 1_{(2)}1'_{(1)}]\otimes[\varepsilon'_{(2)}\otimes 1'_{(2)}]\\
&\quad= [\varepsilon_{(1)}\otimes 1_{(1)}]\otimes[\varepsilon_{(2)}\otimes 1_{(2)}]\otimes[\varepsilon_{(3)}\otimes 1_{(3)}],\\
&([\varepsilon\otimes 1]\otimes\Delta([\varepsilon\otimes 1]))(\Delta([\varepsilon\otimes 1])\otimes[\varepsilon\otimes 1]) =\\
&\quad= [\varepsilon'_{(1)}\otimes 1'_{(1)}]\otimes[\varepsilon_{(1)}\otimes 1_{(1)}][\varepsilon'_{(2)}\otimes 1'_{(2)}]\otimes[\varepsilon_{(2)}\otimes 1_{(2)}]\\
&\quad= [\varepsilon'_{(1)}\otimes 1'_{(1)}]\otimes[\varepsilon'_{(2)}\varepsilon_{(1)}\otimes 1_{(1)}1'_{(2)}]\otimes[\varepsilon_{(2)}\otimes 1_{(2)}]\\
&\quad= [\varepsilon_{(1)}\otimes 1_{(1)}]\otimes[\varepsilon_{(2)}\otimes 1_{(2)}]\otimes[\varepsilon_{(3)}\otimes 1_{(3)}],
\end{aligned}$$

where we used the axioms of a quantum groupoid and the definition of J.

In order to check the axioms (3),(4), let us compute the target counital map ε_t. We have

$$\begin{aligned}
\varepsilon_t([\phi\otimes h]) &= \varepsilon([\varepsilon_{(1)}\otimes 1_{(1)}][\phi\otimes h])[\varepsilon_{(2)}\otimes 1_{(2)}]\\
&= \langle\, \varepsilon_t(1_{(2)}h),\, \phi_{(2)}\varepsilon_{(1)}\,\rangle\langle\, S(1_{(1)}),\, \phi_{(1)}\,\rangle\langle\, 1_{(3)},\, \phi_{(3)}\,\rangle[\varepsilon_{(2)}\otimes 1_{(4)}]\\
&= \langle\, 1'_{(1)}\varepsilon_t(1_{(2)}h),\, \phi_{(2)}\,\rangle\langle\, 1'_{(2)},\, \varepsilon_{(1)}\,\rangle\\
&\qquad \langle\, S(1_{(1)}),\, \phi_{(1)}\,\rangle\langle\, 1_{(3)},\, \phi_{(3)}\,\rangle[\varepsilon_{(2)}\otimes 1_{(4)}]\\
&= \langle\, S(1_{(1)})1'_{(1)}\varepsilon_t(1_{(2)}h)1_{(3)},\, \phi\,\rangle\langle\, 1'_{(2)},\, \varepsilon_{(1)}\,\rangle[\varepsilon_{(2)}\otimes 1_{(4)}]\\
&= \langle\, 1_{(1)}\varepsilon_t(h),\, \phi\,\rangle[\varepsilon\otimes 1_{(2)}].
\end{aligned}$$

Similarly one computes the source counital map :

$$\varepsilon_s([\phi\otimes h]) = [\varepsilon_{(1)}\otimes 1]\langle\, h,\, \varepsilon_t(\phi)S(\varepsilon_{(2)})\,\rangle.$$

Using these formulas we have :

$$\begin{aligned}
&m(\mathrm{id}\otimes S)\Delta([\phi\otimes h]) =\\
&\quad= [\phi_{(1)}\otimes h_{(1)}][S^{-1}(\phi_{(3)})\otimes S(h_{(2)})]\langle\, h_{(2)},\, \phi_{(2)}\,\rangle\langle\, S(h_{(4)}),\, \phi_{(4)}\,\rangle\\
&\quad= [S^{-1}(\phi_{(4)})\phi_{(1)}\otimes h_{(2)}S(h_{(5)})]\langle\, S(h_{(1)}),\, S^{-1}(\phi_{(5)})\,\rangle\\
&\qquad \langle\, h_{(3)},\, S^{-1}(\phi_{(3)})\,\rangle\langle\, h_{(4)},\, \phi_{(2)}\,\rangle\langle\, S(h_{(6)}),\, \phi_{(6)}\,\rangle\\
&\quad= [S^{-1}(\phi_{(3)})\phi_{(1)}\otimes h_{(2)}S(h_{(4)})]\langle\, \varepsilon_t(h_{(3)}),\, S^{-1}(\phi_{(2)})\,\rangle\langle\, h_{(1)}S(h_{(5)}),\, \phi_{(4)}\,\rangle\\
&\quad= [S^{-1}(\phi_{(3)})\phi_{(1)}\otimes 1_{(1)}\varepsilon_t(h_{(2)})]\langle\, 1_{(2)},\, S^{-1}(\phi_{(2)})\,\rangle\langle\, h_{(1)}S(h_{(3)}),\, \phi_{(4)}\,\rangle\\
&\quad= [S^{-1}(\phi_{(3)})\phi_{(1)}\otimes 1_{(1)}1'_{(2)}]\langle\, 1_{(2)},\, S^{-1}(\phi_{(2)})\,\rangle\langle\, 1'_{(1)}\varepsilon_t(h),\, \phi_{(4)}\,\rangle\\
&\quad= [\varepsilon_{(1)}S^{-1}(\phi_{(3)})\phi_{(1)}\otimes 1_{(1)}1'_{(2)}]\langle\, 1_{(2)},\, \varepsilon_{(2)}\,\rangle\langle\, 1'_{(1)}\varepsilon_t(h),\, \phi_{(3)}\,\rangle\\
&\quad= [S^{-1}(\phi_{(2)})\phi_{(1)}\otimes 1'_{(2)}]\langle\, 1'_{(1)}\varepsilon_t(h),\, \phi_{(3)}\,\rangle\\
&\quad= [\varepsilon\otimes 1_{(2)}]\langle\, 1_{(1)}\varepsilon_t(h),\, \phi\,\rangle = \varepsilon_t([\phi\otimes h]),
\end{aligned}$$

and

$$
\begin{aligned}
& m(S\otimes \mathrm{id})\Delta([\phi\otimes h]) = \\
&= [\phi_{(5)}S^{-1}(\phi_{(2)})\otimes S(h_{(3)})h_{(6)}]\langle\, S^2(h_{(4)}),\, \phi_{(4)}\,\rangle\langle\, S(h_{(2)}),\, \phi_{(6)}\,\rangle \\
&\quad \langle\, h_{(1)},\, \phi_{(1)}\,\rangle\langle\, S(h_{(5)}),\, \phi_{(3)}\,\rangle \\
&= [S^{-1}(\varepsilon_{(1)}\varepsilon_t(\phi_{(2)}))\otimes S(h_{(2)})h_{(4)}]\langle\, S(h_{(3)}),\, \varepsilon_{(2)}\,\rangle\langle\, h_{(1)},\, \phi_{(1)}S(\phi_{(3)})\,\rangle \\
&= [S^{-1}(\varepsilon_{(1)}\varepsilon'_{(2)})\otimes S(h_{(2)})h_{(4)}]\langle\, \varepsilon_s(h_{(3)}),\, S(\varepsilon_{(2)})\,\rangle\langle\, h_{(1)}\varepsilon'_{(1)}\varepsilon_t(\phi)\,\rangle \\
&= [S^{-1}(\varepsilon_{(1)}\varepsilon'_{(2)})\otimes \varepsilon_s(h_{(2)})]\langle\, 1_{(1)},\, S(\varepsilon_{(2)})\,\rangle\langle\, h_{(1)}\varepsilon'_{(1)}\varepsilon_t(\phi)\,\rangle \\
&= [\varepsilon_{(3)}S^{-1}(\varepsilon_{(2)})\otimes \varepsilon_s(h_{(2)})]\langle\, h_{(1)},\, \varepsilon_t(\phi)S(\varepsilon_{(2)})\,\rangle \\
&= [\varepsilon_{(1)}\otimes 1]\langle\, h,\, S(\varepsilon_{(2)})\varepsilon_t(\phi)\,\rangle = \varepsilon_s([\phi\otimes h]).
\end{aligned}
$$

In the above computations we used repeatedly the amalgamation relations in $D(H)$:

$$
[\phi\otimes zh] = [(\varepsilon \leftharpoonup z)\phi\otimes h] \quad (z\in H_t), \qquad [\phi\otimes yh] = [(y\rightharpoonup \varepsilon)\phi\otimes h] \quad (y\in H_s)
$$

that follow from (32), the axioms of a quantum groupoid and properties of the counital maps. Finally, we prove the relation which is equivalent to S being both algebra and coalgebra anti-homomorphism:

$$
\begin{aligned}
& m(\mathrm{id}\otimes m)(S\otimes \mathrm{id}\otimes S)(\mathrm{id}\otimes\Delta)\Delta([\phi\otimes h]) = \\
&= m(S\otimes\varepsilon_t)\Delta([\phi\otimes h]) \\
&= S([\phi_{(1)}\otimes h_{(1)}])\varepsilon_t([\phi_{(2)}\otimes h_{(2)}]) \\
&= [S^{-1}(\phi_{(2)})\otimes S(h_{(2)})][\varepsilon\otimes 1_{(2)}] \\
&\quad \langle\, h_{(1)},\, \phi_{(1)}\,\rangle\langle\, S(h_{(3)}),\, \phi_{(3)}\,\rangle\langle\, 1_{(1)}\varepsilon_t(h_{(4)}),\, \phi_{(4)}\,\rangle \\
&= [S^{-1}(\phi_{(2)})\otimes S(h_{(2)})1_{(2)}]\langle\, h_{(1)},\, \phi_{(1)}\,\rangle\langle\, S(h_{(3)})1_{(1)},\, \phi_{(3)}\,\rangle = S([\phi\otimes h]).
\end{aligned}
$$

Note that $D(H)_t = [\varepsilon\otimes H_t]$ and $D(H)_s = [\widehat{H}_s\otimes 1]$. $\square$

Proposition 6.2. *The Drinfeld double $D(H)$ has a quasitriangular structure given by*

$$
\mathcal{R} = \sum_i [\xi^i\otimes 1]\otimes[\varepsilon\otimes f_i], \qquad \bar{\mathcal{R}} = \sum_j [S^{-1}(\xi_j)\otimes 1]\otimes[\varepsilon\otimes f_j] \tag{37}
$$

where $\{f_i\}$ and $\{\xi^i\}$ are dual bases in H and $\widehat{H}$.

Proof. The identities $(\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$ and $(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$ can be written as (identifying $[\widehat{H}^{op}\otimes 1]$ with $\widehat{H}^{op}$ and $[\varepsilon\otimes H]$ with H):

$$
\sum_i \xi^i_{(1)}\otimes\xi^i_{(2)}\otimes f_i = \sum_{ij} \xi^i\otimes\xi^j\otimes f_if_j, \tag{38}
$$

$$
\sum_i \xi^i\otimes f_{i(1)}\otimes f_{i(2)} = \sum_{ij} \xi^j\xi^i\otimes f_j\otimes f_i. \tag{39}
$$

The above equalities can be verified by evaluating both sides on an element $h\in H$ in the third factor (resp., on $\phi\in\widehat{H}^{op}$ in the second factor), see ([M], 7.1.1). To

show that $\mathcal{R}$ is an intertwiner between Δ and Δ^{op}, we compute

$$\begin{aligned}
\mathcal{R}\Delta([\phi\otimes h]) &= \sum_i [\phi_{(1)}\xi^i \otimes h_{(1)}]\otimes[\phi_{(3)}\otimes f_{i(2)}h_{(2)}] \\
&\quad \langle\, S(f_{i(1)}),\, \phi_{(2)}\,\rangle\langle\, f_{i(3)},\, \phi_{(4)}\,\rangle \\
&= \sum_i [\phi_{(1)}S(\phi_{(2)})\xi^i\phi_{(4)}\otimes h_{(1)}]\otimes[\phi_{(3)}\otimes f_i h_{(2)}] \\
&= \sum_i [\xi^i\phi_{(3)}\otimes\langle\, 1_{(1)},\, \varepsilon_t(\phi_{(1)})\,\rangle 1_{(2)}h_{(1)}]\otimes[\phi_{(2)}\otimes f_i h_{(2)}] \\
&= \sum_i [\xi^i\phi_{(2)}\otimes\langle\, 1_{(1)},\, \varepsilon_{(1)}\,\rangle 1_{(2)}h_{(1)}]\otimes[\varepsilon_{(2)}\phi_{(1)}\otimes f_i h_{(2)}] \\
&= \sum_i [\xi^i\phi_{(2)}\otimes h_{(2)}]\otimes[\langle\, \varepsilon_t(h_{(1)}),\, \varepsilon_{(1)}\,\rangle\varepsilon_{(2)}\phi_{(1)}\otimes f_i h_{(3)}] \\
&= \sum_i [\xi^i\phi_{(2)}\otimes h_{(3)}]\otimes[\phi_{(1)}\otimes h_{(1)}S(h_{(2)})f_i h_{(4)}] \\
&= \sum_i [\xi^i_{(2)}\phi_{(2)}\otimes h_{(3)}]\otimes[\phi_{(1)}\otimes h_{(1)}f_i]\langle\, S(h_{(2)}),\, \xi^i_{(1)}\,\rangle\langle\, h_{(4)},\, \xi^i_{(3)}\,\rangle \\
&= \Delta^{op}([\phi\otimes h])\mathcal{R}.
\end{aligned}$$

where we used

$$\sum_i \langle\, a,\, \xi^i\,\rangle f_i = a \quad\text{and}\quad \sum_i \xi^i\langle\, f_i,\, \phi\,\rangle = \phi, \tag{40}$$

for all $a\in H, \phi\in\widehat{H}$. Finally, let us check that the element $\bar{\mathcal{R}} = \sum_j [S^{-1}(\xi_j)\otimes 1]\otimes[\varepsilon\otimes f_j]$ satisfies $\bar{\mathcal{R}}\mathcal{R} = \Delta(1)$ and $\mathcal{R}\bar{\mathcal{R}} = \Delta^{op}(1)$. The first property is equivalent to

$$\sum_{i,j}[\xi^i S^{-1}(\xi^j)\otimes 1]\otimes[\varepsilon\otimes f_j f_i] = [\langle\, 1_{(1)},\, \varepsilon'_{(2)}\,\rangle\varepsilon'_{(1)}\varepsilon_{(1)}\otimes 1]\otimes[\varepsilon\otimes\langle\, 1'_{(1)},\, \varepsilon_{(2)}\,\rangle 1'_{(2)}1_{(2)}],$$

which can be regarded as an equality in $\widehat{H}^{op}\otimes H$:

$$\sum_{i,j}\xi^i S^{-1}(\xi^j)\otimes f_j f_i = \langle\, 1_{(1)},\, \varepsilon'_{(2)}\,\rangle\varepsilon'_{(1)}\varepsilon_{(1)}\otimes\langle\, 1'_{(1)},\, \varepsilon_{(2)}\,\rangle 1'_{(2)}1_{(2)}.$$

Evaluating both sides on arbitrary $\phi\in\widehat{H}$ in the second factor, we get

$$\begin{aligned}
\phi_{(2)}S^{-1}(\phi_{(1)}) &= \langle\, 1_{(1)},\, \varepsilon'_{(2)}\,\rangle\varepsilon'_{(1)}\varepsilon_{(1)}\langle\, 1'_{(1)},\, \varepsilon_{(2)}\,\rangle\langle\, 1'_{(2)}1_{(2)},\phi\,\rangle \\
&= \varepsilon_s^{op}(\phi_{(2)})\varepsilon_s^{op}(\phi_{(1)}),
\end{aligned}$$

where $\varepsilon_s^{op}(\phi) = \phi_{(2)}S^{-1}(\phi_{(1)})$ is the source counital map in $\widehat{H}^{op}$. The second property is similar. □

For the Hopf algebra case the above idea of the proof was proposed in [M], 7.1.1.

Remark 6.3. The dual quantum groupoid $\widehat{D(H)}$ consists of all elements $\sum_k h_k\otimes\phi_k$ in $H\otimes_k\widehat{H}^{op}$ such that

$$\sum_k (h_k\otimes\phi_k)|_J = 0.$$

The structure operations in $\widehat{D(H)}$ are obtained by dualizing those in $D(H)$:

$$\begin{aligned}
(\sum_k h_k \otimes \phi_k)(\sum_l g_l \otimes \psi_l) &= \sum_{k,l} h_k g_l \otimes \phi_k \psi_l, \\
1_{\widehat{D(H)}} &= 1_{(2)} \otimes (\varepsilon \leftharpoonup 1_{(1)}), \\
\Delta(\sum_k h_k \otimes \phi_k) &= \sum_{i,j,k} (h_{k(2)} \otimes \xi^i \phi_{k(1)} \xi^j) \otimes (S(f_i) h_{k(1)} f_j \otimes \phi_{k(2)}), \\
\varepsilon(\sum_k h_k \otimes \phi_k) &= \sum_k \varepsilon(h_k) \widehat{\varepsilon}(\phi_k), \\
S(\sum_k h_k \otimes \phi_k) &= \sum_{i,j,k} f_i S^{-1}(h_k) S(f_j) \otimes \xi^i S(\phi) \xi^j,
\end{aligned}$$

for all $\sum_k h_k \otimes \phi_k$, $\sum_l g_l \otimes \psi_l \in \widehat{D(H)}$, where $\{f_i\}$, $\{\xi^j\}$ are dual bases in $H, \widehat{H}$, respectively.

Proposition 6.4. *The Drinfeld double $D(H)$ is factorizable in the sense of Definition 5.11.*

Proof. First, observe that for any pair of dual bases $\{f_i\}$ and $\{\xi^j\}$ as above and all $g \in H$ and $\psi \in \widehat{H}$ the element

$$Q_{g\otimes\psi} = \sum_{k,l} f_k g f_l \otimes \xi^k \psi S(\xi^l) \in H \otimes_k \widehat{H}^{op}$$

belongs to $\widehat{D(H)}$. Indeed,

$$\begin{aligned}
&\langle\, Q_{g\otimes\psi},\, \phi \otimes zh \,\rangle = \\
&\quad= \langle\, zh_{(1)},\, \phi_{(1)} \,\rangle \langle\, S(h_{(3)}),\, \phi_{(3)} \,\rangle \langle\, g,\, \phi_{(2)} \,\rangle \langle\, h_{(2)}, \psi \,\rangle \\
&\quad= \langle\, h_{(1)},\, (\phi \leftharpoonup z)_{(1)} \,\rangle \langle\, S(h_{(3)}),\, (\phi \leftharpoonup z)_{(3)} \,\rangle \langle\, g,\, (\phi \leftharpoonup z)_{(2)} \,\rangle \langle\, h_{(2)},\, \psi \,\rangle \\
&\quad= \langle\, Q_{g\otimes\psi},\, (\varepsilon \leftharpoonup z)\phi \otimes h \,\rangle,
\end{aligned}$$

for all $h \in H$, $\phi \in \widehat{H}$, $z \in H_t$ and, likewise,

$$\langle\, Q_{g\otimes\psi},\, \phi \otimes yh \,\rangle = \langle\, Q_{g\otimes\psi},\, (y \rightharpoonup \varepsilon)\phi \otimes h \,\rangle.$$

Next, we compute

$$\begin{aligned}
&(Q_{g\otimes\psi} \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R}) = \\
&\quad= \sum_{i,j} Q_{g\otimes\psi}([\xi^j{}_{(2)} \otimes f_{i(2)}])\, [\xi^i \otimes f_j]\, \langle\, S(f_{i(1)}),\, \xi^j{}_{(1)} \,\rangle \langle\, f_{i(3)}, \xi^j{}_{(3)} \,\rangle \\
&\quad= \sum_{i,j} Q_{g\otimes\psi}([\xi^j{}_{(2)} \otimes f_i])\, [S(\xi^j{}_{(1)}) \xi^i \xi^j{}_{(3)} \otimes f_j] \\
&\quad= \sum_{i,j,k,l} [S(\xi^j) \xi^k \psi S(\xi^l) \xi^i \otimes f_j f_k g f_l f_i] \\
&\quad= [\varepsilon'_{(1)} \langle\, 1'_{(2)},\, \varepsilon'_{(2)} \,\rangle \psi \varepsilon_{(1)} \langle\, 1_{(2)},\, \varepsilon_{(2)} \,\rangle \otimes 1'_{(1)} g 1_{(1)}] \\
&\quad= [\psi \varepsilon_{(1)} \langle\, 1_{(2)},\, \varepsilon_{(2)} \,\rangle \otimes g 1_{(1)}],
\end{aligned}$$

where we used the identities

$$\begin{aligned}\sum_j S(\xi^j{}_{(1)})\langle\, g,\, \xi^j{}_{(2)}\,\rangle \otimes \xi^j{}_{(3)} \otimes f_j &= \sum_{i,j} S(\xi^i) \otimes \xi^j \otimes f_i g f_j,\\ \sum_{i,j} S(\xi^i)\xi^j \otimes f_i f_j &= \varepsilon_{(1)} \otimes 1_{(1)}\langle\, 1_{(2)},\, \varepsilon_{(2)}\,\rangle,\end{aligned}$$

that follow from (38), (39), (40) and from axioms (3),(4) of a quantum groupoid. Therefore,

$$\begin{aligned}&\langle\, 1_{(2)},\, \varepsilon_{(2)}\,\rangle Q_{g1_{(1)}\otimes\psi\varepsilon_{(1)}}(\mathcal{R}_{21}\mathcal{R}) =\\ &= [\psi\varepsilon_{(1)}\varepsilon'_{(1)} \otimes g1'_{(1)}1_{(1)}]\langle\, 1_{(2)},\, \varepsilon'_{(2)}\,\rangle\langle\, 1'_{(2)},\, \varepsilon_{(2)}\,\rangle\\ &= [\varepsilon_{(1)} \otimes 1_{(1)}][\psi \otimes g]S([\varepsilon_{(2)} \otimes 1_{(2)}]) = \mathrm{Ad}\,^l_1([\psi \otimes g]),\end{aligned}$$

Thus, we conclude from Lemma 2.2 that the map

$$\widehat{D(H)} \ni x \mapsto (Q_x \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R}) \in C_{D(H)}(D(H)_s)$$

is surjective, i.e., $D(H)$ is factorizable. □

7. Ribbon quantum groupoids

Definition 7.1. A ribbon quantum groupoid is a quasitriangular quantum groupoid H with an invertible central element $\nu \in H$ such that

$$\Delta(\nu) = \mathcal{R}_{21}\mathcal{R}(\nu \otimes \nu) \quad \text{and} \quad S(\nu) = \nu. \tag{41}$$

The element ν is called a *ribbon element* of H.

For an object V of Rep(H) we define the twist $\theta_V : V \to V$ to be the multiplication by ν :

$$\theta_V(v) = \nu \cdot v, \quad v \in V. \tag{42}$$

Proposition 7.2. *Let $(H, \mathcal{R}, \nu)$ be a ribbon quantum groupoid. The family of homomorphisms $\{\theta_V\}_V$ is a twist in the braided monoidal category Rep(H) compatible with duality. Conversely, if $\theta_V(v) = \nu \cdot v$ with $\nu \in H$ is a twist in Rep(H), then ν is a ribbon element of H.*

Proof. Since ν is an invertible central element of H, the homomorphism θ_V is an H-linear isomorphism. The twist identity $c_{W,V}c_{V,W}(\theta_V \otimes \theta_W) = \theta_{V\otimes W}$ follows from the properties of ν:

$$c_{W,V}c_{V,W}(\theta_V \otimes \theta_W)(x) = \mathcal{R}_{21}\mathcal{R}(\nu \cdot x^{(1)} \otimes \nu \cdot x^{(2)}) = \Delta(\nu) \cdot x = \theta_{V\otimes W}(x),$$

for all $x = x^{(1)} \otimes x^{(2)} \in V \otimes W$. Clearly, the identity $\mathcal{R}_{21}\mathcal{R}(\nu \otimes \nu) = \Delta(\nu)$ is equivalent to the twist property. It remains to prove that

$$(\theta_V \otimes \mathrm{id}_{V^*})b_V(z) = (\mathrm{id}_V \otimes \theta_{V^*})b_V(z),$$

for all $z \in H_t$, i.e., that

$$\sum_i \nu z_{(1)} \cdot \xi^i \otimes z_{(2)} \cdot f_i = \sum_i z_{(1)} \cdot \xi^i \otimes \nu z_{(2)} \cdot f_i,$$

where $\sum_i \xi^i \otimes f_i$ is the canonical element in $V^* \otimes V$. Evaluating the first factors of the above equality on an arbitrary $v \in V$, we get the equivalent condition :

$$\sum_i (\nu z_{(1)} \cdot \xi^i)(v) z_{(2)} \cdot f_i = \sum_i (z_{(1)} \cdot \xi^i)(v) \nu z_{(2)} \cdot f_i,$$

which reduces to $z_{(2)}S(\nu z_{(1)})\cdot v = S(z_{(1)})\nu z_{(2)}\cdot v$. The latter easily follows from the centrality of $\nu = S(\nu)$ and properties of H_t. □

Proposition 7.3. *The category Rep(H) is a ribbon category if and only if H is a ribbon quantum groupoid.*

Proof. Follows from Propositions 4.2, 5.2, and 7.2. □

For any endomorphism f of an object V of Rep(H), we define, following ([T2], I.1.5), its *quantum trace*

$$\mathrm{tr}_q(f) = d_V c_{V,V^*}(\theta_V f\otimes \mathrm{id}_{V^*})b_V \tag{43}$$

with values in End(H_t) and the *quantum dimension* of V by $\dim_q(V) = \mathrm{tr}_q(\mathrm{id}_V)$. The next lemma gives an explicit computation of tr_q and $\dim_q$ via the usual trace of endomorphisms.

Proposition 7.4. *Let $(H,\mathcal{R},\nu)$ be a ribbon quantum groupoid, f be an endomorphism of an object V in Rep(H). Then*

$$tr_q(f)(z) = Tr(S(1_{(1)})u\nu f)z1_{(2)}, \qquad dim_q(V)(z) = Tr(S(1_{(1)})u\nu)z1_{(2)}, \tag{44}$$

where Tr is the usual trace of endomorphisms, and $u\in H$ is the Drinfeld element.

Proof. Since the trace of an endomorphism $h\in \mathrm{End}_k(H_t)$, in terms of the canonical element $\sum_i f_i\otimes\xi^i \in V\otimes_k V^*$, is $\mathrm{Tr}(h) = \sum_i \xi^i(h(f_i))$, the definition of tr_q gives:

$$\begin{aligned}
\mathrm{tr}_q(f)(z) &= d_V c_{V,V^*}(\theta_V f\otimes \mathrm{id}_{V^*})b_V(z)\\
&= d_V(\sum_i \mathcal{R}^{(2)}z_{(2)}\cdot\xi^i\otimes\mathcal{R}^{(1)}\nu z_{(1)}\cdot f(f_i))\\
&= \sum_i(\mathcal{R}^{(2)}z_{(2)}\cdot\xi^i)(1_{(1)}\mathcal{R}^{(1)}\nu z_{(1)}\cdot f(f_i))1_{(2)}\\
&= \sum_i \xi^i(S(\mathcal{R}^{(2)}z_{(2)})1_{(1)}\mathcal{R}^{(1)}\nu z_{(1)}\cdot f(f_i))1_{(2)}\\
&= \mathrm{Tr}(S(1_{(1)})u\nu f)z1_{(2)},
\end{aligned}$$

where we used formulas (24) and (25) defining b_V and d_V. □

Corollary 7.5. *Let k be algebraically closed. If H-module H_t is irreducible (which happens exactly when $H_t\cap Z(H) = k$, i.e., when H is connected (*[N]*, 3.11,* [BNSz]*, 2.4), then $tr_q(f)$ and $dim_q(V)$ are scalars:*

$$tr_q(f) = (\dim H_t)^{-1}\ Tr(u\nu f), \qquad \dim_q(V) = (\dim H_t)^{-1}\,Tr(u\nu). \tag{45}$$

Proof. An endomorphism of an irreducible module is multiplication by a scalar, therefore, we must have $\mathrm{Tr}(S(1_{(1)})u\nu f)1_{(2)} = \mathrm{tr}_q(f)(1) = \mathrm{tr}_q(f)1$. Applying the counit to both sides and using that $\varepsilon(1) = \dim H_t$, we get the result. □

8. Towards modular categories

Let us first recall some definitions needed in this section. Let $\mathcal{V}$ be a ribbon *Ab*-category over k, i.e., such that all Hom(V,W) are k-vector spaces (for all objects $V,W\in\mathcal{V}$) and both operations $\circ$ and $\otimes$ are k-bilinear.

An object $V\in\mathcal{V}$ is said to be *simple* if any endomorphism of V is multiplication by an element of k. We say that a family $\{V_i\}_{i\in I}$ of objects of $\mathcal{V}$ dominates an object V of $\mathcal{V}$ if there exists a finite set $\{V_{i(r)}\}_r$ of objects of this family (possibly,

with repetitions) and a family of morphisms $f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}$ such that $\mathrm{id}_V = \sum_r f_r g_r$.

A modular category ([T2], II.1.4) is a pair consisting of a ribbon *Ab*-category $\mathcal{V}$ and a finite family $\{V_i\}_{i\in I}$ of simple objects of $\mathcal{V}$ satisfying four axioms:

(i) There exists $0 \in I$ such that V_0 is the unit object.

(ii) For any $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to V_i^*.

(iii) All objects of $\mathcal{V}$ are dominated by the family $\{V\}_{i\in I}$.

(iv) The square matrix $S = \{S_{ij}\}_{i,j\in I} = \{\mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i})\}_{i,j\in I}$ is invertible over k (here tr_q is the quantum trace in a ribbon category defined by (43)).

If a quantum groupoid H is connected and semisimple over an algebraically closed field, modularity of $\mathrm{Rep}(H)$ is equivalent to $\mathrm{Rep}(H)$ being ribbon and such that the matrix $S = \{S_{ij}\}_{i,j\in I} = \{\mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i})\}_{i,j\in I}$, where I is the set of all (equivalent classes of) irreducible representations, is invertible.

Remark 8.1. Recall that $h \in H$ is a left (resp. right) *integral* if $xh = \varepsilon_t(x)h$ (resp. $hx = h\varepsilon_s(x)$) for all $x \in H$ ([BNSz], 3.24). A *Haar integral* is a two-sided integral h which is normalized, i.e., $\varepsilon_t(h) = \varepsilon_s(h) = 1$. Existence of a Haar integral in a quantum groupoid H is equivalent to H being semisimple and possessing an invertible element g such that $S^2(x) = gxg^{-1}$ for any $x \in H$ and $\chi(g^{-1}) \neq 0$ for all irreducible characters χ of H ([BNSz], 3.27).

The following lemma extends a result known for Hopf algebras ([EG], 1.1).

Lemma 8.2. *Let H be a connected, ribbon, factorizable quantum groupoid over an algebraically closed field k, and assume that H has a Haar integral. Then Rep(H) is a modular category.*

Proof. Note that H is semisimple by Remark 8.1. We only need to prove the invertibility of the matrix formed by

$$\begin{aligned} S_{ij} &= \mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i}) \\ &= (\dim H_t)^{-1}\mathrm{Tr}((u\nu) \circ c_{V_i,V_j} \circ c_{V_j,V_i}) \\ &= (\dim H_t)^{-1}(\chi_j \otimes \chi_i)((u\nu \otimes u\nu)\mathcal{R}_{21}\mathcal{R}), \end{aligned}$$

where V_i are as above, $I = \{1, \dots n\}$, $\{\chi_j\}$ is a basis in the space $C(H)$ of characters of H (we used above the formula (45) for the quantum trace).

Observe that the linear map $F : \phi \mapsto (\phi \otimes \mathrm{id})(\mathcal{R}_{21}\mathcal{R})$ takes any element of the form $\phi = \chi \leftharpoonup u\nu$ (i.e., $\phi(h) = \chi(u\nu h)\ \forall h \in H$), where $\chi \in C(H)$, into $Z(H)$. Indeed, for any such ϕ and all $\psi \in \widehat{H}$, $h \in H$ we have, using the fact that $u \in C_H(H_s)$ (this follows from Lemma 5.3), the properties of ε_s and ε_s^{op}, the relation $\Delta^{op}(h)\mathcal{R} = \Delta(h)\mathcal{R}$ $(h \in H)$, and the centrality of χ :

$$\begin{aligned} \langle\, F(\phi)h,\, \psi\,\rangle &= \langle\, u\nu\mathcal{R}^{(2)}\mathcal{R}'^{(1)},\, \chi\,\rangle\langle\, \mathcal{R}^{(1)}\mathcal{R}'^{(2)}h,\, \psi\,\rangle \\ &= \langle\, u\nu\mathcal{R}^{(2)}\mathcal{R}'^{(1)}\varepsilon_s^{op}(h_{(1)}),\, \chi\,\rangle\langle\, \mathcal{R}^{(1)}\mathcal{R}'^{(2)}h_{(2)},\, \psi\,\rangle \\ &= \langle\, u\nu\mathcal{R}^{(2)}h_{(2)}\mathcal{R}'^{(1)}S^{-1}(h_{(1)}),\, \chi\,\rangle\langle\, \mathcal{R}^{(1)}h_{(3)}\mathcal{R}'^{(2)},\, \psi\,\rangle \\ &= \langle\, u\nu h_{(2)}\mathcal{R}^{(2)}\mathcal{R}'^{(1)}S^{-1}(h_{(1)}),\, \chi\,\rangle\langle\, h_{(3)}\mathcal{R}^{(1)}\mathcal{R}'^{(2)},\, \psi\,\rangle \\ &= \langle\, u\nu S(h_{(1)})h_{(2)}\mathcal{R}^{(2)}\mathcal{R}'^{(1)},\, \chi\,\rangle\langle\, h_{(3)}\mathcal{R}^{(1)}\mathcal{R}'^{(2)},\, \psi\,\rangle \\ &= \langle\, u\nu\mathcal{R}^{(2)}\mathcal{R}'^{(1)},\, \chi\,\rangle\langle\, h\mathcal{R}^{(1)}\mathcal{R}'^{(2)},\, \psi\,\rangle \\ &= \langle\, hF(\phi),\, \psi\,\rangle, \end{aligned}$$

therefore $F(\phi) \in Z(H)$. Since H is factorizable, we know from Corollary 5.12 that the restriction

$$F : \{\phi \in \widehat{H} \mid \phi = \phi \circ \mathrm{Ad}\,_1^r\} \to C_H(H_s)$$

is a linear isomorphism. Since $\chi \leftharpoonup u\nu$ belongs to the subspace on the left hand side, we have a linear isomorphism between $C(H) \leftharpoonup u\nu$ and $Z(H)$, hence, there exists an invertible matrix $T = (T_{ij})$ representing the map F in the bases of $C(H)$ and $Z(H)$, i.e., such that $F(\chi_j \leftharpoonup u\nu) = \sum_i T_{ij} e_i$. Then

$$\begin{aligned} S_{ij} &= (\dim H_t)^{-1} \chi_i(u\nu F(\chi_j \leftharpoonup u\nu)) = (\dim H_t)^{-1} \sum_k T_{kj} \chi_i(u\nu e_k) \\ &= (\dim H_t)^{-1} (\dim V_i) \chi_i(u\nu) T_{ij}. \end{aligned}$$

Therefore, $S = DT$, where $D = \mathrm{diag}\{(\dim H_t)^{-1}(\dim V_i)\chi_i(u\nu)\}$. If g is an element from Remark 8.1 then $u^{-1}g$ is an invertible central element of h and $\chi_i(u^{-1}) \neq 0$ for all χ_i. By Corollary 5.9 $uS(u) = c$ is invertible central, therefore $\chi_i(u) = \chi_i(c)\chi_i(S(u^{-1}))) \neq 0$. Hence, $\chi_i(u\nu) \neq 0$ for all i and D is invertible. □

Example 8.3. An example of a modular category can be constructed from *elementary* quantum groupoids classified in [NV1], 3.2. A quantum groupoid H is called *elementary* if $H \cong M_n(k)$. Then it is determined, up to an isomorphism, by one of its counital subalgebras

$$H_t \cong \oplus_\alpha M_{n_\alpha}(k),$$

$n_\alpha,\ \alpha = 1 \dots N$ are positive integers, $n = \sum_\alpha n_\alpha^2$. From this one can see that

$$D(H) = [\widehat{H}^{op} \otimes H] = [\widehat{H}_s \widehat{H}_t \otimes H][\varepsilon \otimes H] = H,$$

where for subsets $A \subset \widehat{H}^{op}, b \subset H$ we set $[A \otimes B] = \{[a \otimes b] \in D(H) | a \in A, b \in B\}$. Hence H is the Drinfeld double of itself. The R-matrix of H is

$$R = \sum_{i,j,k,l\alpha} \frac{1}{n_\alpha} E^{ik\alpha}_{jl\alpha} \otimes E^{kl\alpha}_{ij\alpha},$$

where $\{E^{kl\beta}_{ij\alpha}\}_{i,j=1\dots n_\alpha}$ is a system of matrix units in H. Both the Drinfeld and the ribbon elements are equal to 1. Thus, all the conditions of Lemma 8.2 are satisfied, the category $\mathrm{Rep}(H)$ is modular with a unique irreducible object.

9. C^*-quantum groupoids and unitary modular categories

Definition 9.1. A $*$-*quantum groupoid* is a quantum groupoid over a field k with involution, whose underlying algebra H is equipped with an antilinear involutive algebra anti-homomorphism $* : H \to H$ such that $\Delta \circ * = (* \otimes *)\Delta$. A $*$-quantum groupoid is said to be C^*-*quantum groupoid*, if $k = \mathbb{C}$ and H is a finite-dimensional C^*-algebra, i.e., $x^*x = 0$ if and only if $x = 0$, $\forall x \in H$.

Definition 9.1 together with the uniqueness of the unit, counit and antipode imply that

$$1^* = 1, \quad \varepsilon(h^*) = \overline{\varepsilon(h)}, \quad (S \circ *)^2 = \mathrm{id}$$

for all h in a $*$-quantum groupoid H. It is also easy to check the relations

$$\varepsilon_t(h)^* = \varepsilon_t(S(h)^*),\ \varepsilon_t(h)^* = \varepsilon_t(S(h)^*),$$

therefore, H_t and H_s are $*$-subalgebras. The dual $\widehat{H}$ is also a $*$-quantum groupoid with respect to the $*$-operation

$$\langle \phi^*, h \rangle = \overline{\langle \phi, S(h)^* \rangle} \qquad \text{for all } \phi \in \widehat{H},\ h \in H. \tag{46}$$

The square of the antipode of a C^*-quantum groupoid is an inner automorphism, i.e., $S^2(h) = ghg^{-1}$ for some $g \in H$. It is easy to see that there is a unique such g satisfying the following conditions ([BNSz], 4.4):

(i) $\mathrm{tr}(\pi_\alpha(g^{-1})) = \mathrm{tr}(\pi_\alpha(g)) \neq 0$ for all irreducible representations π_α of H (here tr is the usual trace on a matrix algebra);
(ii) $S(g) = g^{-1}$, and
(iii) $\Delta(g) = (g \otimes g)\Delta(1) = \Delta(1)(g \otimes g)$.

This element g is called the *canonical group-like element* of H.

Remark 9.2. (i) Any C^*-quantum groupoid satisfies the conditions of Remark 8.1, so it always possess a Haar integral.

(ii) If H is a C^*-quantum groupoid, then its dual $\widehat{H}$ is also a C^*-quantum groupoid (see [BNSz], 4.5, [NV1], 2.3.10).

Groupoid algebras and their duals give examples of commutative and cocommutative C^*-quantum groupoids if the ground field $k = \mathbb{C}$ (in which case $g^* = g^{-1}$ for all $g \in G$).

One can check that for a quasitriangular $*$-quantum groupoid $\bar{\mathcal{R}} = \mathcal{R}^*$.

Proposition 9.3. *If H is a C^*-quantum groupoid, then $D(H)$ is a quasitriangular C^*-quantum groupoid.*

Proof. First let us show that $\widehat{D(H)}$, equipped with a natural involution

$$\langle X^*,\, \phi \otimes h \rangle = \sum_k \overline{\langle g_k,\, (S\phi)^* \rangle \langle S(h)^*,\, \psi_k \rangle},$$

where $X = \sum_k g_k \otimes \psi_k \in \widehat{D(H)}, h \in H,\, \phi \in \widehat{H}$, is a C^*-subalgebra of the tensor product C^*-algebra $H \otimes \widehat{H}^{op}$. For this it suffices to show that $X^*|_J = 0$, i.e., $\langle X^*,\, \phi \otimes zh \rangle = \langle X^*,\, (\varepsilon \leftharpoonup z)\phi \otimes h \rangle$, $\langle X^*,\, \phi \otimes yh \rangle = \langle X^*,\, (y \rightharpoonup \varepsilon)\phi \otimes h \rangle$ for $z \in H_t, y \in H_s$. For instance, one computes:

$$\begin{aligned} \langle X^*,\, \phi \otimes zh \rangle &= \sum_k \overline{\langle g_k,\, (S\phi)^* \rangle \langle S(z)^* S(h)^*,\, \psi_k \rangle}, \\ \langle X^*,\, (\varepsilon \leftharpoonup z)\phi \otimes h \rangle &= \sum_k \overline{\langle g_k,\, S((\varepsilon \leftharpoonup z)\phi)^* \rangle \langle S(h)^*,\, \psi_k \rangle}, \end{aligned}$$

for all $z \in H_t$. The right-hand sides of the above equations are equal since

$$\overline{\langle z,\, \varepsilon_{(1)} \rangle} S(\varepsilon_{(2)})^* = \langle S(z)^*, \varepsilon_{(2)} \rangle \varepsilon_{(1)}.$$

Similarly one gets the other relation.

To prove that the comultiplication of $\widehat{D(H)}$ is a $*$-homomorphism we compute

$$\begin{aligned} \Delta(X)^* &= \sum_{i,j,k} (g_{k}{}^*_{(2)} \otimes \xi^{j*} \psi_{k}{}^*_{(1)} \xi^{i*}) \otimes (f_j^* g_{k}{}^*_{(1)} S(f_i)^* \otimes \psi_{k}{}^*_{(2)}) \\ &= \sum_{i,j,k} (g_{k}{}^*_{(2)} \otimes \xi^i \psi_{k}{}^*_{(1)} \xi^j) \otimes (S(f_i) g_{k}{}^*_{(1)} f_j \otimes \psi_{k}{}^*_{(2)}) = \Delta(X^*), \end{aligned}$$

where we use that $\sum_j (\xi^j)^* \otimes S(f_j)^* = \sum_j \xi^j \otimes f_j$ for every pair of dual bases. Thus, $\widehat{D(H)}$ is a C^*-quantum groupoid and so is $D(H)$ (see Remark 9.2). $\square$

In [EG] it was shown that a quasitriangular semisimple Hopf algebra is automatically ribbon with ribbon element $\nu = u^{-1}$, where u is the Drinfeld element. We are able to get a similar result for C^*-quantum groupoids.

Proposition 9.4. *A quasitriangular C^*-quantum groupoid H is automatically ribbon with ribbon element $\nu = u^{-1}g = gu^{-1}$, where u is the Drinfeld element from Definition 5.8 and g is the canonical group-like element implementing S^2.*

Proof. Since u also implements S^2 (Proposition 5.7), $\nu = u^{-1}g$ is central, therefore $S(\nu)$ is also central. Clearly, u must commute with g. The same Proposition gives $\Delta(u^{-1}) = \mathcal{R}_{21}\mathcal{R}(u^{-1} \otimes u^{-1})$, which allows us to compute

$$\Delta(\nu) = \Delta(u^{-1})(g \otimes g) = \mathcal{R}_{21}\mathcal{R}(u^{-1}g \otimes u^{-1}g) = \mathcal{R}_{21}\mathcal{R}(\nu \otimes \nu).$$

Propositions 5.6 and 5.7 and the trace property imply that

$$\begin{aligned} \mathrm{tr}(\pi_\alpha(u^{-1})) &= \mathrm{tr}(\pi_\alpha(\mathcal{R}^{(2)} S^2(\mathcal{R}^{(1)}))) \\ &= \mathrm{tr}(\pi_\alpha(S^3(\mathcal{R}^{(1)}) S(\mathcal{R}^{(2)})) = \mathrm{tr}(\pi_\alpha(S(u^{-1}))). \end{aligned}$$

Since $u^{-1} = \nu g^{-1}$ and ν is central, the above relation means that

$$\mathrm{tr}(\pi_\alpha(\nu))\mathrm{tr}(\pi_\alpha(g^{-1})) = \mathrm{tr}(\pi_\alpha(S(\nu)))\mathrm{tr}(\pi_\alpha(g)),$$

and, therefore, $\mathrm{tr}(\pi_\alpha(\nu)) = \mathrm{tr}(\pi_\alpha(S(\nu)))$ for any irreducible representation π_α, which shows that that $\nu = S(\nu)$. $\square$

Corollary 9.5. *For a connected ribbon C^*-quantum groupoid H we have:*

$$tr_q(f) = (\dim H_t)^{-1}\, Tr_V(g \circ f), \qquad \dim_q(V) = (\dim H_t)^{-1}\, Tr_V(g).$$

for any $f \in End(V)$, where V is an H-module.

To define the (unitary) representation category $\mathrm{URep}(H)$ of a C^*-quantum groupoid H we consider *unitary* H-modules, i.e., H-modules V equipped with a scalar product

$$(\cdot,\cdot) : V \times V \to \mathbb{C} \qquad \text{such that} \qquad (h \cdot v, w) = (v, h^* \cdot w)\ \forall h \in H, v, w \in V.$$

The notion of a morphism in this category remains the same as in $\mathrm{Rep}(H)$. The monoidal product of $V, W \in \mathrm{URep}(H)$ is defined as follows. We construct a tensor product $V \otimes_{\mathbb{C}} W$ of Hilbert spaces and remark that the action of $\Delta(1)$ on this left H-module is an orthogonal projection. The image of this projection is, by definition, the monoidal product of V, W in $\mathrm{URep}(H)$. Clearly, this definition is compatible with the monoidal product of morphisms in $\mathrm{Rep}(H)$.

For any $V \in \mathrm{URep}(H)$, the dual space V^* is naturally identified $(v \to \overline{v})$ with the conjugate Hilbert space, and under this identification we have $h \cdot \overline{v} = \overline{S(h)^* \cdot v}$ $(v \in V, \overline{v} \in V^*)$. In this way V^* becomes a unitary H-module with scalar product $(\overline{v}, \overline{w}) = (w, gv)$, where g is the canonical group-like element of H.

The unit object in $\mathrm{URep}(H)$ is H_t equipped with scalar product $(z,t)_{H_t} = \varepsilon(zt^*)$ (it is known [BNSz], [NV1] that the restriction of ε to H_t is a non-degenerate positive form). One can verify that the maps l_V, r_V and their inverses are isometries. For example, let us show that the adjoint map for l_V is exactly its inverse. We have:

$$(l_V(1_{(1)} \cdot z \otimes 1_{(2)} \cdot v), w) = (z \cdot v, w)\ (\forall z \in H_t, v, w \in V).$$

On the other hand:

$$\begin{aligned}
(1_{(1)}\cdot z\otimes 1_{(2)}\cdot v, l_V^{-1}w) &= (zS(1_{(1)})\otimes 1_{(2)}\cdot v, S(1_{(1)})\otimes 1_{(2)}\cdot w)\\
&= \varepsilon(zS(1_{(1)})S(1_{(1)})^*)(1^*_{(2)}1_{(2)}\cdot v, w) = (z\cdot v, w).
\end{aligned}$$

Proposition 4.2 implies that URep(H) is a monoidal category with duality (see also [BNSz], Section 3).

Remark 9.6. a) One can check that for a quasitriangular $*$-quantum groupoid the braiding is an isometry in URep(H): $c_{V,W}^{-1} = c_{V,W}^*$.

b) For a ribbon C^*-quantum groupoid H, the twist is an isometry in URep(H). Indeed, the relation $\theta_V^* = \theta_V^{-1}$ is equivalent to the identity $S(u^{-1}) = u^*$, which follows from Proposition 5.6 and Remark 9.6a).

A *Hermitian ribbon category* over the field k with involution is an *Ab*-ribbon category over k endowed with a conjugation of morphisms $f\mapsto \overline{f}$ satisfying natural conditions (see [T2], II.5.2):

$$\overline{\overline{f}} = f,\qquad \overline{f+g}=\overline{f}+\overline{g},\qquad \overline{cf}=\overline{c}\,\overline{f}\quad (c\in k), \tag{47}$$

$$\overline{f\otimes g} = \overline{f}\otimes\overline{g},\quad \overline{f\circ g}=\overline{g}\circ\overline{f},\qquad \overline{c_{V,W}} = (c_{V,W})^{-1}, \overline{\theta_V}=\theta_V^{-1}, \tag{48}$$

$$\overline{b}_V = d_V\circ c_{V,V^*}(\theta_V\otimes id_{V^*}),\qquad \overline{d}_V = (id_{V^*}\otimes\theta_V^{-1})c^{-1}_{V^*,V}\circ b_V. \tag{49}$$

A *unitary ribbon category* is a Hermitian ribbon category over the field $\mathbb{C}$ such that for any morphism f we have $\mathrm{tr}_q(f\overline{f})\geq 0$.

In a natural way we have a *conjugation* of morphisms in URep(H). Namely, for any morphism $f: V\to W$ we define $\overline{f}: W\to V$ as $\overline{f}(w) = \overline{f^*(\overline{w})}$ for any $w\in W$. Here $\overline{w}\in W^*$, $f^*: W^*\to V^*$ is the standard dual of f (see [T2], I.1.8) and $\overline{f^*(\overline{w})}\in V$.

Lemma 9.7. *Given a quasitriangular C^*-quantum groupoid H, URep(H) is a unitary ribbon Ab-category with respect to the above conjugation of morphisms.*

Proof. Relations (47) are obvious, (48) follows from Remarks 9.6.

Let us prove relations (49). On the one hand, for all $v\in V, \phi\in V^*, z\in H_t$ we have, using the definitions of d_V, c_{V,V^*}, θ_V, Propositions 5.6, 5.7 and the notation $\omega_{v,\phi}(L) = (Lv,\phi)$ for a linear operator L and two vectors v,ϕ of a Hilbert space:

$$\begin{aligned}
&(d_V\circ c_{V,V^*}(\theta_V\otimes id_{V^*})(v\otimes\phi), z)_{H_t} =\\
&\quad= (d_V\circ c_{V,V^*}(gu^{-1}v\otimes\phi), z)_{H_t}\\
&\quad= (d_V[S(\mathcal{R}^{*(2)})\phi\otimes\mathcal{R}^{(1)}gu^{-1}v], z)_{H_t}\\
&\quad= \varepsilon[(1_{(1)}\mathcal{R}^{(1)}gu^{-1}v, S(\mathcal{R}^{*(2)})\phi)1_{(2)}z^*]\\
&\quad= (\omega_{v,\phi}\otimes\varepsilon)[(S(\mathcal{R}^{(2)})\otimes 1)\Delta(1)(\mathcal{R}^{(1)}gu^{-1}\otimes z^*)]\\
&\quad= \omega_{v,\phi}[S(\mathcal{R}^{(2)})S(z^*)\mathcal{R}^{(1)}gu^{-1}]\\
&\quad= \omega_{v,\phi}[S(\mathcal{R}^{(2)})\mathcal{R}^{(1)}z^*gu^{-1}] = (z^*gv,\phi).
\end{aligned}$$

And, on the other hand, using the definition of b_V, we compute :

$$\begin{aligned}(\overline{b}_V(v\otimes\phi),z)_{H_t} &= (v\otimes\phi,\sum_i z_{(1)}f_i\otimes S(z_{(2)})^*\xi^i)_{V\otimes V^*}\\ &= \sum_i (v,z_{(1)}f_i)(S(z_{(2)})^*\xi^i,g\phi)\\ &= \sum_i (z^*_{(1)}v,f_i)(\xi^i,S(z_{(2)})g\phi)\\ &= (z^*_{(1)}v,S(z_{(2)})g\phi)=(v,zG\phi)=(z^*gv,\phi),\end{aligned}$$

whence the first part of (49) follows. To establish the second part, note that for all $v\in V,\phi\in V^*,z\in H_t$ we have, using the definitions of b_V,c_{V,V^*},θ_V, Propositions 5.6, 5.7 and the properties of ν:

$$\begin{aligned}&((\mathrm{id}_{V^*}\otimes\theta_V^{-1})c^{-1}_{V^*,V}\circ b_V(z),\phi\otimes v)_{V^*\otimes V}=\\ &= ((\mathrm{id}_{V^*}\otimes\theta_V^{-1})c^{-1}_{V^*,V}\sum_i z_{(1)}f_i\otimes S(z_{(2)})^*\xi^i),\phi\otimes v)_{V^*\otimes V}\\ &= ((\mathrm{id}_{V^*}\otimes\theta_V^{-1})\sum_i S(\mathcal{R}^{*(1)})^*S(z_{(2)})^*\xi^i\otimes\mathcal{R}^{*(2)}z_{(1)}f_i),\phi\otimes v)_{V^*\otimes V}\\ &= \sum_i(\phi,gS(\mathcal{R}^{*(1)})^*S(z_{(2)})^*\xi^i)(\nu^{-1}\mathcal{R}^{*(2)}z_{(1)}f_i,v)\\ &= \sum_i(S(z_{(2)})S(\mathcal{R}^{*(1)})g\phi,\xi^i)(f_i,\nu z^*_{(1)}\mathcal{R}^{(2)}v)\\ &= (S(z_{(2)})S(\mathcal{R}^{*(1)})\phi,\nu z^*_{(1)}\mathcal{R}^{(2)}v)=(\mathcal{R}^{*(2)}zS(\mathcal{R}^{*(1)})g\phi,\nu v)\\ &= ((S^{-1}(\mathcal{R}^{(1)})\mathcal{R}^{(2)})^*S(z)g\phi,\nu v)=(S^{-1}(u)^*S(z)g\phi,\nu v)=(\phi,S(z^*)v).\end{aligned}$$

On the other hand, using the definition of d_V, we obtain:

$$\begin{aligned}(\overline{d}_V(z),\phi\otimes v)_{V^*\otimes V} &= (z,d_V(\phi\otimes v))_{H_t}=(z,(1_{(1)}v,\phi)1_{(2)})_{H_t}\\ &= \overline{\varepsilon((1_{(1)}v,\phi)1_{(2)}z^*)}=\overline{(\omega_{v,\phi}\otimes\varepsilon)(\Delta(1)(1\otimes z^*))}\\ &= \overline{(\omega_{v,\phi}(S(z^*))}=(\phi,S(z^*)v).\end{aligned}$$

The condition $\mathrm{tr}_q(f\overline{f})=\mathrm{Tr}(gff^*)\geq 0$ for any morphism f follows from Remark 9.6b) and from the positivity of g. $\square$

The next proposition extends ([EG], 1.2).

Theorem 9.8. *If H is a connected C^*-quantum groupoid, then URep$(D(H))$ is a unitary modular category.*

Proof. The proof follows from Lemmas 9.7, 8.2 and Propositions 6.4, 9.3. $\square$

10. Appendix

Here we collected some results on ribbon and modular quantum groupoids which extend the corresponding facts for Hopf algebras. **1.** There is a procedure analogous to ([RT1], 3.4), that extends any quasitriangular quantum groupoid $(H,\mathcal{R},\bar{\mathcal{R}})$ to a ribbon quantum groupoid in a canonical way. For this we need

Lemma 10.1 (cf. ([RT1], 3.3)). *(i) A ribbon element ν satisfies*

$$\varepsilon_t(\nu)=\varepsilon_s(\nu)=1\quad\textit{and}\quad\nu^2=(vu)^{-1},$$

where u and v are the elements defined in Proposition 5.7.
(ii) If ν_1 and ν_2 are two ribbon elements of $(H, \mathcal{R})$, then $\nu_2 = E\nu_1$, where $E \in H$ is an invertible central element such that $E = S(E) = E^{-1}$, $\Delta(E) = \Delta(1)(E \otimes E)$ (i.e., E is group-like), and $\varepsilon_t(E) = \varepsilon_s(E) = 1$.

Proof. (i) The definition of counit implies :

$$\begin{aligned} \nu &= (\mathrm{id} \otimes \varepsilon)\Delta(\nu) = \nu \mathcal{R}^{(2)} {\mathcal{R}'}^{(1)} \varepsilon(\mathcal{R}^{(1)} {\mathcal{R}'}^{(2)} \nu) \\ &= \nu \mathcal{R}^{(2)} {\mathcal{R}'}^{(1)} \varepsilon(\varepsilon_s(\mathcal{R}^{(1)}) {\mathcal{R}'}^{(2)} \nu) = \nu {\mathcal{R}'}^{(1)} \varepsilon(\varepsilon_s({\mathcal{R}'}^{(2)}) \nu) \\ &= \nu S(1_{(2)}) \varepsilon(1_{(1)} \nu) = \nu S(\varepsilon_t(\nu)), \end{aligned}$$

hence $\varepsilon_t(\nu) = 1$. We used here the identity $\varepsilon(hg) = \varepsilon(\varepsilon_s(h)g)$, $h, g \in H$ and Lemma 5.6. Similarly, $\varepsilon_s(\nu) = 1$. Using the antipode property, we compute

$$\begin{aligned} 1 &= \varepsilon_t(\nu) = m(\mathrm{id} \otimes S)\Delta(\nu) \\ &= \mathcal{R}^{(2)} {\mathcal{R}'}^{(1)} S({\mathcal{R}'}^{(2)}) S(\mathcal{R}^{(1)}) \nu^2 \\ &= v S^2(\mathcal{R}^{(2)}) S(\mathcal{R}^{(1)}) \nu^2 = vu\nu^2. \end{aligned}$$

(ii) Set $E = \nu_1^{-1} \nu_2$. Then E is central and invertible, $S(E) = E$, and from part (i) we conclude that $E^2 = 1$. Next,

$$\Delta(E) = \bar{\mathcal{R}} \bar{\mathcal{R}}_{21} (\nu_1^{-1} \otimes \nu_1^{-1}) \mathcal{R}_{21} \mathcal{R} (\nu_2 \otimes \nu_2) = \Delta(1)(E \otimes E).$$

Applying the counit to both sides of the last equality, we get $E = E\varepsilon_t(E) = E\varepsilon_s(E)$, i.e., $\varepsilon_t(E) = \varepsilon_s(E) = 1$. □

Proposition 10.2. *Let $\tilde{H} = H + H\nu$ be a central extension of H, consisting of formal linear combinations $h + g\nu$ with $h, g \in H$. Then $(\tilde{H}, \mathcal{R}, \nu)$ is a ribbon quantum groupoid with operations*

$$\begin{aligned} (h + g\nu)(h' + g'\nu) &= (hh' + gg'(vu)^{-1}) + (hg' + gh')\nu, \\ \Delta(h + g\nu) &= \Delta(h) + \Delta(g) \mathcal{R}_{21} \mathcal{R} (\nu \otimes \nu), \\ \varepsilon(h + g\nu) &= \varepsilon(h) + \varepsilon(g), \\ S(h + g\nu) &= S(h) + S(g)\nu. \end{aligned}$$

Note that $\tilde{H}$ contains $H = \{h + 0\nu \mid g \in H\}$ as a quantum subgroupoid.

Proof. One verifies that Δ is a homomorphism exactly as in [RT1]. The properties of $\mathcal{R}$ and ν follow directly from definitions. For the counit axiom we have, using the properties of counital maps, Proposition 5.6, and Lemma 10.1 :

$$\begin{aligned} (\varepsilon \otimes \mathrm{id})\Delta(h + g\nu) &= h + \varepsilon(g_{(1)} \nu \varepsilon_t(\mathcal{R}^{(2)} {\mathcal{R}'}^{(1)})) g_{(2)} \nu \mathcal{R}^{(1)} {\mathcal{R}'}^{(2)} \\ &= h + g_{(2)} S(\varepsilon_s(g_{(1)} \nu)) = h + g\nu, \\ (\mathrm{id} \otimes \varepsilon)\Delta(h + g\nu) &= h + \mathcal{R}^{(2)} {\mathcal{R}'}^{(1)} g_{(1)} \nu \varepsilon(\varepsilon_s(\mathcal{R}^{(1)} {\mathcal{R}'}^{(2)}) g_{(2)} \nu) \\ &= h + S(\varepsilon_t(g_{(2)} \nu)) g_{(1)} \nu = h + g\nu. \end{aligned}$$

Axioms (2) and (1) of Definition 2.1 can be verified by a direct computation.

Next, we observe that $\varepsilon_t(h+g\nu) = \varepsilon_t(h)+\varepsilon_t(g)$ and $\varepsilon_s(h+g\nu) = \varepsilon_s(h)+\varepsilon_s(g)$. The antipode axiom follows from the identity $\nu^2 vu = 1$ provided by Lemma 10.1:

$$\begin{aligned} m(\mathrm{id}\otimes S)\Delta(h+g\nu) &= \varepsilon_t(h) + g_{(1)}\mathcal{R}^{(2)}\mathcal{R}'^{(1)}S(\mathcal{R}'^{(2)})S(\mathcal{R}^{(1)})S(g_{(2)})\nu^2 \\ &= \varepsilon_t(h) + g_{(1)}vuS(g_{(2)})\nu^2 = \varepsilon_t(h+g\nu), \\ m(S\otimes \mathrm{id})\Delta(h+g\nu) &= \varepsilon_s(h) + S(g_{(1)})S(\mathcal{R}^{(1)})S(\mathcal{R}'^{(2)})\mathcal{R}'^{(1)}\mathcal{R}^{(2)}g_{(2)}\nu^2 \\ &= \varepsilon_s(h) + S(g_{(1)})vug_{(2)}\nu^2 = \varepsilon_s(h+g\nu). \end{aligned}$$

The anti-multiplicative properties of the antipode follow from the facts that $S(uv) = uv$ and $S(\nu) = \nu$. ☐

2. Let us establish a relation between modular quantum groupoids and modular categories. A morphism $f : V \to W$ in a ribbon *Ab*-category $\mathcal{V}$ is said to be *negligible* if for any morphism $g : W \to V$ we have $\mathrm{tr}(fg) = 0$. $\mathcal{V}$ is said to be *pure* if all negligible morphisms in this category are equal to zero. A purification procedure transforming any ribbon *Ab*-category into a pure ribbon *Ab*-category is described in ([T2], XI.4.2); this procedure transforms hermitian ribbon *Ab*-categories into hermitian pure ribbon *Ab*-categories ([T2], XI.4.3). We say that a family $\{V\}_{i\in I}$ of objects of $\mathcal{V}$ quasidominates an object V of $\mathcal{V}$ if there exists a finite set $\{V_{i(r)}\}_r$ of objects of this family (possibly, with repetitions) and a family of morphisms $f_r : V_{i(r)} \to V, g_r : V \to V_{i(r)}$ such that $\mathrm{id}_V - \sum_r f_r g_r$ is negligible. If $\mathcal{V}$ is pure, then quasidomination coincides with domination. Let $(H, \mathcal{R}, \nu)$ be a ribbon quantum groupoid. Then an H-module V of finite k-rank is said to be *negligible* if $\mathrm{tr}_q(f) = 0$ for any $f \in \mathrm{End}(V)$. If k is algebraically closed, then any irreducible H-module is a simple object of $\mathrm{Rep}(H)$.

Definition 10.3. A modular quantum groupoid consists of a ribbon quantum groupoid $(H, \mathcal{R}, \nu)$ together with a finite family of simple H-modules of finite rank $\{V\}_{i\in I}$ such that:

(i) for some $0 \in I$, we have $V_0 = H_t$, the unit object of $\mathrm{Rep}(H)$;
(ii) for each $i \in I$, there exists $i^* \in I$ such that V_{i^*} is isomorphic to V_i^*;
(iii) for any $k, l \in I$, the tensor product $V_k \otimes V_l$ splits as a finite direct some of certain $\{V\}_{i\in I}$ (possibly with multiplicities) and a negligible H-module;

To formulate the last condition, let $S_{i,j} = \mathrm{tr}_q(c_{V_i,V_j} \circ c_{V_j,V_i}),\ i, j \in I$, where the braiding c_{V_i,V_j} was defined in 5.1 and the quantum trace tr_q in 7.4.

(iv) The square matrix $[S_{i,j}]_{i,j\in I}$ is invertible in $M_{|I|}(k)$.

For any modular quantum groupoid, we define a subcategory $\mathcal{C}$ of $\mathrm{Rep}(H)$ as follows. The objects of $\mathcal{C}$ are H-modules of finite rank quasidominated by $\{V\}_{i\in I}$ and morphisms are H-morphisms of such modules; all the operations in $\mathcal{C}$ are induced by the corresponding operations in $\mathrm{Rep}(H)$. Now taking into account the results of the previous sections and repeating the proof of ([T2], XI.5.3.2), we have the first statement of the following

Proposition 10.4. *If $(H, \mathcal{R}, \nu, \{V\}_{i\in I})$ is modular, then the subcategory $(\mathcal{C}, \{V\}_{i\in I})$ of Rep(H) is quasimodular in the sense of (*[T2]*, XI.4.3). Conversely, if $(\mathcal{C}, \{V\}_{i\in I})$ is quasimodular, then $(H, \mathcal{R}, \nu, \{V\}_{i\in I})$ is modular.*

The proof of the second statement follows directly from the comparison of [T2], XI.4.3 and the above definition of a modular quantum groupoid. Purifying $(\mathcal{C}, \{V\}_{i\in I})$ as in ([T2], XI.4.2), we get a modular category ([T2], II.1.4).

References

[BNSz] G. Böhm, F. Nill and K. Szlachányi, *Weak Hopf algebras I. Integral theory and C^*-structure*, J. Algebra, **221** (1999), 385–438.

[BSz1] G. Böhm, K. Szlachányi, *A coassociative C^*-quantum group with nonintegral dimensions*, Lett. in Math. Phys, **35**, (1996), 437–456.

[BSz2] G. Böhm, K. Szlachányi, *Weak Hopf algebras II. Representation theory, dimensions and the Markov trace*, `math.QA/9906045` (1999), to appear in J. Algebra.

[BSz3] G. Böhm, K. Szlachányi, *Weak C^*-Hopf algebras and multiplicative isometries*, `math.QA/9810070` (1998), to appear in J. Operator Theory.

[D] V.G. Drinfeld, *Quantum groups*, Proc. of ICM Berkeley 1986. Providence, R.I. vol. 1 (1987), 798–820.

[EG] P. Etingof and S. Gelaki, *Some properties of finite dimensional semisimple Hopf algebras*, Math. Res. Letters **5** (1998), 191–197.

[EK] D. Evans and Y. Kawahigashi, *On Ocneanu's theory of asymptotic inclusions for subfactors, topological quantum field theories and quantum doubles*, Int. J. Math.**6** (1995), 205–228.

[EN] P. Etingof, D. Nikshych, *Dynamical quantum groups at roots of* 1 `math.QA/0003221` (2000).

[GHJ] F. Goodman, P. de la Harpe, and V.F.R. Jones, *Coxeter Graphs and Towers of Algebras*, M.S.R.I. Publ. **14**, Springer, Heidelberg, 1989.

[KS] A. Klimyk and K. Schmüdgen, *Quantum Groups and Their Representations*, Springer Verlag (1997).

[M] S. Majid, *Foundations of quantum group theory*, Cambridge University Press, Cambridge, (1995).

[N] D. Nikshych, *Duality for actions of weak Kac algebras and crossed product inclusions of II_1 factors*, to appear in the Journal of Operator Theory, `math.QA/9810049` (1998).

[NV1] D. Nikshych, L. Vainerman, *Algebraic versions of a finite dimensional quantum groupoid*, Lecture Notes in Pure and Appl. Math. **209** (2000), 189-221.

[NV2] D. Nikshych, L. Vainerman, *A characterization of depth* 2 *subfactors of II_1 factors*, J. Func. Analysis **171** (2000), no. 2, 278-307.

[NV3] D. Nikshych, L. Vainerman, *A Galois correspondence for II_1 factors*, `math.QA/0001020` (2000).

[NV4] D. Nikshych, L. Vainerman, *Finite quantum groupoids and their applications.* `math.QA/0006057` (2000).

[Ni] F. Nill, *Axioms of Weak Bialgebras* `math.QA/9805104` (1998).

[NSzW] F. Nill, K. Szlachányi, H.-W. Wiesbrock, *Weak Hopf algebras and reducible Jones inclusions of depth 2*, I : From crossed products to Jones towers. *math.QA/9806130* (1998).

[O] A. Ocneanu, *Quantum Cohomology, Quantum groupoids and Subfactors*, unpublished talk given at the First Caribbean School of Mathematics and Theoretical Physics, Guadeloupe, (1993).

[P] R. Pierce, *Associative algebras*, Graduate Texts in Mathematics, **88**. Springer Verlag, (1982).

[R] J. Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Math. **793**, Springer-Verlag, 1980.

[RT1] N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127**(1990), 1-26.

[RT2] N. Reshetikhin, V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math. **103**(1991), 547-597.

[S] W. Szymański, *Finite index subfactors and Hopf algebra crossed products*, Proc. Amer. Math. Soc. **120** (1994), no. 2, 519–528.

[T1] V. Turaev, *Modular categories and 3-manifold invariants*, Int. J. of Modern Phys. B **6**(1992), 1807–1824.

[T2] V. Turaev, *Quantum invariants of knots and 3-manifolds*, de Gruyter Studies in Mathematics, **18**, Walter de Gruyter & Co., Berlin, (1994).

[W] Y. Watatani, *Index for C^*-subalgebras*, Memoirs AMS **424** (1990).

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